An Introduction to Hawking Effect and Black Hole Information Problem

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I. INTRODUCTION

One of the main problems of the fundamental physics is to find a theory of quantum gravity. one motivation to study such a theory is to understand the earliest moments of the universe, where we expect that quantum effects are dominant. In the search for this theory, it is better to consider simpler problems. A simpler problem involves black holes. They also contain a singularity in their interior.
In the 1975 Hawking showed that black holes behave as thermal objects[1]. they have a temperature that leads to Hawking raditation. they also have an entropy given by the size of the horizon. Hawking objected to this idea through what we now know as the "Hawking information paradox."[2] He argued that a black hole would destroy quantum information, and that the Von Neumann entropy of the universe would increase by the process of black hole formation and evaporation.
Since then finding a proper solution for this paradox has been one of the main possible hints for finding a theory of quantum gravity.
In this article I review the principles of Hawking radiation, following Hawking's original calculations. Then some of concepts of information theory would be reviewed and with all the ingredients, I restate Hawkings argument in regard of the information paradox. At the end I will mention some of the recent developments in regard of this paradox. For further discussion, reader can consult to more thorough reviews[3][4][5].
Throughout this article I will work in units with $G = c = \hbar = 1$ unless explicitly stated otherwise.

II. HAWKING EFFECT

A. Cauchy Problem in General Relativity

To better understand the process of Hawking Radiation, it would be beneficial to review the initial value problem in context of General Relativity. For further discussion on this topic one can refer to Wald(1984) or Reall[6, 7].

Definition: Let (M, g) be a time-orientable spacetime. A partial *partial Cauchy surface* Σ is a hypersurface for which no two points are connected by a causal curve in M. the *future domain of dependence* of Σ, denoted $D^+(\Sigma)$, is the set of $p \in M$ such that every past-inextendible causal curve through p intersects Σ. The past domain of dependence, $D^-(\Sigma)$, is defined similarly. the *domain of dependence* of M is $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.

The significance of these surfaces comes from the fact that hyperbolic differential equations such as Einstein's equations allow unique solutions in $D(\Sigma)$ given certain initial data on Σ (for a proof, refer to Ringström(2009)[8]). I close this subsection with another definition.

Definition: A spacetime (M, g) is *globally hyperbolic* if it admits a Cauchy surface: a partial Cauchy surface Σ such that $M = D(\Sigma)$.

B. Black Hole Mechanics

Now I summarize the classical black hole mechanics in three theorems (the quantities used in the following theorems are discussed in the subsection II.C.2.):
0) The zeroth law states that the surface gravity κ of a black hole is constant on horizon.
1) The first law states that variations in mass M, area A, angular momentum L and charge Q of a black hole obey [9]

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta L - \nu\delta Q, \tag{1}$$

where Ω is the angular velocity of the horizon and ν is the difference in the electrostatic potential between infinity and the horizon.
2) The second law is the area theorem [10] proved by Hawking in 1971. The area of a black hole is nondecreasing in time,

$$\delta A \geq 0. \tag{2}$$

This result assumes that the spacetime is globally hyperbolic and that the dominant energy condition holds.

The resemblance of the laws above and those of the thermodynamics raised the question - is this identification more than formal? initially, there was little reason to believe that these had anything to do with 'real' thermodynamics. This changed with Hawking's discovery that, when general relativity is coupled to quantum field theory, black holes have a temperature

$$T = \hbar\frac{\kappa}{2\pi}. \tag{3}$$

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Our task in the next subsection would be to review Hawking's argument and discovery of the temperature above.

C. Particle Creation by Black Holes

The basis of Hawking's argument in [1], is the idea of semiclassical gravity, in which one can quantize fields quantum mechanically, but keep gravity classical. Hence, one considers quantum field theory in a fixed curved background.

1. Quantum Fields in Curved Background

In the case of QFTs, it seems reasonable to follow the usual approach in formulating gravitational field theories and make use of the so called "Minimal coupling" principle to write the field expansions in the general covariant form. Indeed this approach works, minus a number of difficulties that are caused due to the loss of the concept of "Global families of inertial observers". this issue is more or less the same issue that one would face when trying to make the transition between the special and general theory of relativity.
Perhaps the most important difference to keep in mind is to remember that the concept of particles is frame dependent in curved space, while in the flat space, the Lorentz invariance confirms the frame independence of particles. for the case of scalar field theory in the curved spacetime, we can demonstrate this issue rather simply. One can choose a complete basis f_w of solutions of the curved scalar wave equation $g^{ab}\nabla_a\nabla_b\phi = 0$. We choose this basis so that the basis functions are delta function orthonormal $(f_w, f_{w'}) = \delta(w-w^{'})$ with respect to the inner product

$$(f,h) = -i\int_\Sigma (f\nabla_n h^* - h\nabla_n f^*)\sqrt{|\gamma|}d^{n-1}x, \quad (4)$$

where the integral is taken over a Cauchy surface and n is the normal to that surface and γ is the metric induced on it.
The quantum field ϕ can be expanded in the basis as

$$\phi = \int dw(a_w f_w + a^\dagger_w f^*_w), \quad (5)$$

where a_w and $a^\dagger_w$ are operators satisfying

$$[a_{w'}, a^\dagger_w] = \delta(w^{'} - w), \quad [a_{w'}, a_w] = [a^\dagger_{w'}, a^\dagger_w] = 0. \quad (6)$$

Finally to fully specify the theory we would define the vacuum states corresponding to a_w,

$$a_w \left|0\right\rangle_a = 0, \quad (7)$$

for $\forall w > 0$. One could express the field theory in another arbitrary basis of solutions $\{p_w, p^*_w\}$ equivalently by the expansion

$$\phi = \int dw(b_w p_w + b^\dagger_w p^*_w), \quad (8)$$

where the analogous commutator expressions to that of a_ws would apply to b_ws. The vacuum state for this basis is similarly given by

$$b_w \left|0\right\rangle_b = 0, \quad (9)$$

for $\forall w > 0$. Any two basis of solutions are related to each other by the so called *Bogolubov transformations*

$$\begin{aligned} p_w &= \int dw^{'}(\alpha_{ww'} f_{w'} + \beta_{ww'} f^*_{w'}) \\ f_w &= \int dw^{'}(\alpha^*_{w'w} p_{w'} - \beta_{w'w} p^*_{w'}), \end{aligned} \quad (10)$$

where $\alpha_{ww'}$ and $\beta_{ww'}$ are called *Bogolubov coefficients*. Then one can readily check that expansions operators are related by

$$b_w = \int dw^{'}(\alpha^*_{ww'} a_{w'} - \beta^*_{ww'} a^\dagger_{w'}). \quad (11)$$

With the tools above we can finally calculate the following quantity

$${}_a\left\langle 0\right| (N^b_w) \left|0\right\rangle_a = {}_a\left\langle 0\right| (b^\dagger_w b_w) \left|0\right\rangle_a = \int dw^{'} |\beta_{ww'}|^2, \quad (12)$$

which states that a vacuum state for a certain basis need not to have zero number of particles for another basis, or in other words the concept of particles is frame dependent.

2. Black Holes

In this subsection I briefly review some of concepts about black holes. A stationary black hole spacetime has a killing vector X^a which is normal to the horizon, and has norm $X^a X_a = 0$ on the horizon (for a proof refer to Hawking and Ellis[11]). the surface gravity κ is defined by $\nabla^b(X^a X_a) = -2\kappa X^b$ on the horizon. the horizon area A is the area of the intersection of the horizon with a constant time slice, which is a 2-sphere in all of the cases considered here.
According to Birkhoff's theorem, the Schwarzchild metric below is the unique spherical symmetric solution to the einstein's vacuum equation $R_{ab} = 0$,

$$ds^2 = -(1-2M/r)dt^2 + \frac{1}{1-2M/r}dr^2 + r^2 d\Omega^2. \quad (13)$$

Here $d\Omega^2$ is the metric of a unit 2-sphere. The spacetime has an event horizon where the norm of the timelike killing vector ∂_t vanishes. in the coordinates above horizon lies at $r = 2M$ and has area $A = 4\pi M^2$. the parameter M is the ADM mass of the spacetime. Finally

one can show that $r=0$ is a curvature singularity.
Expressions for the metric in different coordinates are usefull for future calculations,

$$\begin{aligned} ds^2 &= (1-2M/r)(-dt^2+dr^{*2})+r^2d\Omega^2 \\ &= -\frac{2M}{r}e^{-r/2M}e^{(v-u)/4M}dudv+r^2d\Omega^2 \\ &= -\frac{32M^3}{r}e^{-r/2M}dUdV+r^2d\Omega^2, \end{aligned} \tag{14}$$

where the relations between different coordinates are

$$\begin{aligned} dr^* &= \frac{dr}{1-2M/r}, \quad r^*=r+2Mln(|2M/r-1|) \\ u &= t-r^*, \quad v=t+r^* \\ U &= -e^{-u/(4M)}, \quad V=e^{v/(4M)}. \end{aligned} \tag{15}$$

3. *Particle Emission*

Using the black hole geometry and the QFT in curved spacetime introduced in the past subsections, one can consider the collapsing matter with empty vacuum states in the early times and with the use of eq.(12) show that the expected number of late time *out* particles with frequency w_i is

$$_{in}\langle 0|\, b_i^\dagger b_i\, |0\rangle_{in} = \frac{\Gamma_i}{(e^{2w_i\pi/\kappa}-1)}, \tag{16}$$

where Γ_i is a greybody factor, which can be thought of as arising from backscattering of wavepackets off of the gravitational field and into the black hole. The complete calculation of the distribution above can be found in the Appendix below.
Eq.(16) is a black body or thermal spectrum, with temprature

$$T=\hbar\frac{\kappa}{2\pi}. \tag{17}$$

One fascinating implication is that the classical black hole mechanics theorems and the laws of thermodynamics have more than a formal analogy. According to eq.(1), a black hole radiates with temprature $T=\hbar\frac{\kappa}{2\pi}$, and has an entropy

$$S_{bh}=\frac{1}{4}A. \tag{18}$$

The entropy above is referred to as the *Bekenstein-Hawking entropy*. Reinstating units we have

$$S_{bh}=\frac{c^3A}{4G\hbar}. \tag{19}$$

The second law of black hole mechanics now states that S_{bh} is non-decreasing classically. But it does decrease quantum mechanically by Hawking radiation: the black hole loses energy by emitting radiation and therefore gets smaller. However, this radiation itself has entropy and the total entropy $S_{bh}+S_{matter}$ does not decrease. This is a special case of the generalized second law due to Bekenstein[12], which states that the total entropy $S=S_{bh}+S_{matter}$, is non-decreasing in any physical process.
Hawking also calculated particle production in quantum fields by charged and rotating black holes. Calculations have also been done for emission of fermions and gravitons and linearized perturbations of the metric. In all of these cases one finds a thermal spectrum,

$$< N_w^{bh} >= \frac{\Gamma_w}{e^{\frac{2\pi(w-\mu)}{\hbar\kappa}}\pm 1}, \tag{20}$$

where the +1 corresponds to fermions and -1 to bosons. In thermodynamics, μ is called a chemical potential.
To be truly thermal, one must check that there are no hidden correlations in the observed particles. In his 1976 paper[2] Hawking showed that there are indeed no correlations and distribution is truly thermal.

III. BLACK HOLE EVAPORATION

The energy of the Hawking radiation must come from the black hole itself. Hawking's calculation neglects the effect of the radiation on the spacetime geometry. An accurate calculation of this backreaction would involve quantum gravity. However, one can estimate the rate of mass loss by using Stefan's law for the rate of energy loss by a blackbody:

$$\frac{dE}{dt}\approx -\alpha AT^4, \tag{21}$$

where α is a dimensionless constant and we approximate Γ_i by treating the black hole as perfectly absorbing sphere of area A (roughly the black hole horizon area) in Minkowski spacetime. Plugging in $E=M$ with $A\propto M^2$ and $T\propto 1/M$ gives $dM/dt\propto -1/M^2$. Hence the black hole evaporates away completely in a time

$$\tau\sim M^3\sim 10^{71}(\frac{M}{M_\odot})^3 sec. \tag{22}$$

This is a very crude calculation but it is expected to be a reasonable approximation at least until the size of the black hole becomes comparable to the Planck mass (1 in our units), when quantum gravity effects are expected to become important. This process of *black hole evaporation* leads to the information paradox, which will be discussed in section V.

IV. ENTROPY AND INFORMATION

One of the central aspects of the information paradox is the concept of information conservation. In both classical and quantum mechanics there is a very precise sense in which information is never lost from a closed isolated system. In classical mechanics this is a consequence of the Hamilton's equations of motion and Liouville's theorem. In quantum mechanics, this conservation is expressed as the unitarity of the time evolution of a closed system.

A precise definition of information is provided by the concept of entropy. For a probability distribution $\rho(p,q)$ in phase space the mean information in the system is given by

$$S = -\int dqdp\rho(p,q)log\rho(p,q). \tag{23}$$

But for our purposes the quantum mechanical version of this entropy is of more importance. For a quantum mechanical system given by a density matrix ρ, the Von Neumann entropy defined by

$$S = -tr(\rho ln\rho), \tag{24}$$

characterizes the entropy of the system.

In classical physics, the only reason for introducing a phase space distribution is a lack of detailed knowledge of the state. However in quantum mechanics there is also another reason, entanglement. Entanglement refers to the quantum correlations between the system under investigation and a second system. For a pure state system $\rho_{AB} = |\psi\rangle\langle\psi|$ in the full Hilbert space $H_A \otimes H_B$ the Von Neumann entropy vanishes and one can prove that for subsystems in H_A and H_B one has

$$S(\rho_A) = S(\rho_B). \tag{25}$$

There's is no reason for the states in the subsystems to be pure. In fact the only possible pure configuration for them is the product state, which in turn results the vanishing entropy for each subsystem. Thus one can measure the entanglement of the these two subsystems by $S_E = S(\rho_A) = S(\rho_B)$ which differs from zero for entangled states.

The entropy defined in eq.s (23) and (24) is referred to as the *fine grained entropy*. The entropy in the context of thermodynamics is the so called *coarse grained* or *thermal* entropy defined to be the sum of the entropies of the small subsystems:

$$S_{Thermal} = \sum_i S_i. \tag{26}$$

In the definition above the usual choice of coarse graining is the one that puts the subsystems in thermal states $\rho_i = \frac{e^{-\beta H_i}}{Z_i}$. In the course of the evolution of the system the fine grained entropy is conserved while the coarse grained entropy can only increase, which is the second law of thermodynamics.

V. INFORMATION PARADOX

Now we consider a system whose state is pure and its evolution is unitary. For such systems, following the calculations done by Page[13], entanglement entropy has a certain behavior.

instead of following Page's direct calculations, to understand his results, we consider the simple example of a box with perfectly reflecting walls. Inside the box we have a bomb which can explode and fill the box with radiation. The box has a small hole that allows the thermal radiation to slowly leak out. The entire system Σ consists of the subsystem B that includes everything in the box. The subsystem A consists of everything outside of the box, in this case, outgoing photons.

Initially the bomb is in its ground state, and B has vanishing entropy. When the bomb explodes, it fill the box with thermal radiation. The coarse grained entropy of the box increases, but its fine grained entropy does not. Furthermore, no photons have yet escaped, so $S(A) = 0$ at this time. So we have

$$S_{Thermal}(B) \neq 0, \;\; S(A) = S(B) = 0 \tag{27}$$

Next, photons slowly leak out. The result is that the interior and exterior of the box become entangled. The entanglement entropy, which is equal for A and B, begins to increase. The thermal entropy in the box decreases:

$$S_{Entanglement} \neq 0, \;\; S_{Thermal}(A) \neq 0, \;\; S_{Thermal}(B) \neq 0 \tag{28}$$

Eventually, all of the photons escape the box. The course grained entropy as well as the fine grained entropy in the box tends to zero. The box is in a pure state; its ground state.

At this time, the coarse grained entropy of the exterior radiation has increased to its final value. The second law of thermodynamics insures that $S_{Thermal}(A)$ is larger than $S_{Thermal}(B)$ just after the explosion. But the fine grained entropy of A must vanish, since the entanglement has gone to zero.

A physical notion of information mainly due to Page is the following,

$$I = S_{Thermal} - S. \tag{29}$$

It can be thought of as the hidden subtle correlations between subsystems that make the state of Σ pure. With the definition above for the box example there is a point at which $S_{Thermal}(A) = S_{Thermal}(B)$ that defines the time at which the information in the outside radiation begins to grow. Before that point, a good deal of energy has escaped, but no information. the time at which this emergence of information happens is called the *Page time*. Figure 1 demonstrates the plot of various entropies discussed above. Thus we see how information conservation works for a conventional quantum system. The consequence of this principle is that the final radiation field outside the box must be in a pure state.

Hawking in his 1976 paper[2] argued that if we consider the initial state of the black hole and its surroundings to be a pure state, then due to evaporation after a certain time (eq.(22)), system would only consist of thermal radiation and thus the purity would not be restored. In

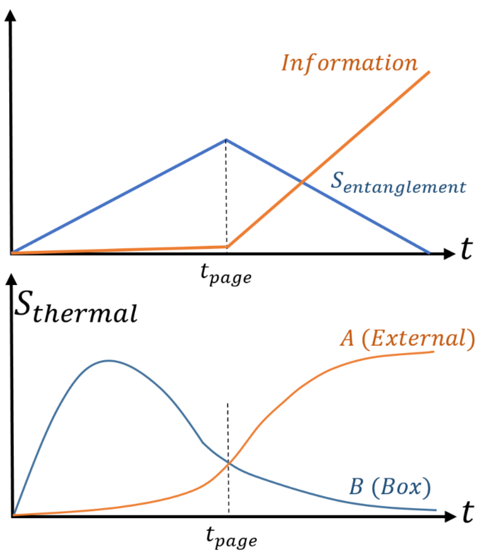

FIG. 1. Plots showing various entropies and information in the box example.

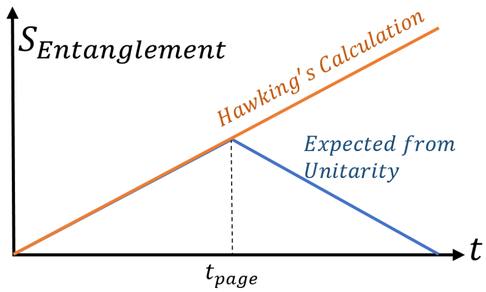

FIG. 2. Schematic behavior of the entropy of the the outgoing radiation. The precise shape of the lines depends on the black hole and the details of the matter fields being radiated. In orange we see Hawking's result, the entropy monotonically increases until t_{end}, when the black hole completely evaporates. If the process is unitary, then it should follow the so called Page curve, denoted in blue.

fact, the only possibilities would seem to be that either information is lost during the entire process of formation and evaporation, or the information is restored to the outside world at the very end of the evaporation process, when quantum gravity is at effect. However, we have seen above that after the Page time, the entanglement entropy of the system must decrease, so by the time the black hole has small mass and entropy, the entanglement entropy of the radiation cannot be larger than the black hole's remaining entropy. This comes from the fact that at the Page time black hole is still very big and semi-classical calculations done by Hawking should still work. Thus, even if all information were emitted at the very end of the evaporation process, a law of nature would be violated from the viewpoint of the external observer. The situation is even worse if the information is not emitted at all. A final possibility that was advocated by some authors is that black holes never completely evaporate. Instead they end their lives as stable Planck-mass remnants that contain all the lost information. Obviously such remnants would have to have an enormous, or even infinite entropy. Such objects would be extremely pathological.

There are two more possibilities worth pointing out. One is that the horizon is not penetrable. which means that a freely falling observer would encounter a "brick wall" just above the horizon. The reason that this was never seriously entertained, especially by relativists, is that it badly violates the equivalence principle. Since the near horizon region of a Schwarzschild black hole is essentially flat spacetime, any violent disturbance to a freely falling system would violate the glorified equivalence principle.

Finally there is a last possibility. The information conservation principle requires all information to be returned to the outside in Hawking radiation. The equivalence principle, on the other hand, requires information to freely pass through the horizon. What if both of them could happen?! this possibility is excluded by the so called "No cloning theorem" in quantum mechanics, which prevents one from copying information and sending duplicates inside and outside the horizon.

So with all the discussion above one concludes that if the unitarity of evolution in quantum mechanics is true, then the information lost in the process of evaporation is a paradox. This is the so called "Hawking information paradox."

VI. FURTHER DEVELOPMENTS

Here I review some of the more recent developments in regard of the information paradox. perhaps the most prominent result after Hawking's argument would be the AdS/CFT correspondence by Maldacena[14] expressing the correspondence of the gravitational theories with unitary conformal field theories. This result strongly suggests that in course of the evaporation, information is not lost and somewhere in the argument we are making a mistake.

Another important result was published in the famous AMPS paper(2012)[15] showing that either information is lost in the course of evaporation or there is a "Firewall" near the horizon of the black hole, which exchanges the entanglement between the ingoing radiation and the early radiation. loss of information is mostly considered not to be correct due to AdS/CFT but firewall hypothesis is a matter of debate.

Recent developments with quantum extremal surfaces and holography, such as islands[16] suggest that neither information is lost nor there is a firewall. They instead argue that what happens is that the interior and outside of horizon are connected in higher dimensions.

VII. APPENDIX: CALCULATIONS OF PARTICLE EMISSION

In this appendix I follow Hawking's original approach for deriving the Hawking effect. First we remember that in the case of spherical symmetry one can decompose the solutions of the wave equation using the so called spherical harmonic functions

$$\phi_{wlm}(t, r^*, \Omega) = \psi(r^*)Y_{lm}(\Omega)e^{-twt}, \tag{30}$$

so that the wave equation reduces to the radial equation

$$\begin{gathered}(\partial_t^2 - \partial_{t^*}^2 + W(r))\psi e^{-twt} = 0, \\ W(r) = (1 - 2M/r)(2M/r^3 + l(l+1)/r^2).\end{gathered} \tag{31}$$

Note that in terms of the tortoise coordinate, horizon lies at $r^* \to -\infty$, whereas in the asymptotically flat limit $r \to \infty$, we also have $r^* \to \infty$. in the latter region, the potential behaves as $W(r) \to \frac{l(l+1)}{r^2}$, and near the horizon, we have $W(r) \to e^{r^*/2M}$. Therefore in these two limits, the solutions ϕ_{wlm} are plane waves in $t \pm r^*$, i.e. plane waves in u, v. using these we can write our field expansions like the expansions we mentioned in subsection II.C.1. . for notational simplicity we drop angular indices further on.

Following Hawking's argument in [1], we do the calculations for a black hole formed by a gravitational collapse. The idea is that in the far past, spacetime is nearly Minkowski, the largest gravitational effects being at the surface of the star, and we can assume that the quantum state is empty of the so called *in*-particles near I^- (the past null infinity). We will call this state $|0\rangle_{in}$. The star collapses to form a black hole. Hawking found that near I^+ (the future null infinity), the state $|0\rangle_{in}$ contains a thermal flux of *out*-particles. The particles produced are known as Hawking radiation.

The conformal diagram in figure 3, demonstrates the spacetime of a collapsing star. one can easily check that I^- is a Cauchy surface. We will take the early time positive frequency modes to be the solutions f_w to the wave equation that behave near I^- like

$$f_w(u, v) \to e^{-iwv}. \tag{32}$$

Far from the star spacetime becomes flat and v becomes an ingoing null coordinate for flat space wave equation. Therefore these modes are same as usual Minkowski modes one would usually use in QFT in flat space.

Now following the eq.(5) we define the field expansion operators and write the field expansion,

$$\phi(u, v) = \int dw(a_w f_w + a_w^\dagger f_w^*). \tag{33}$$

The vacuum state in then taken to satisfy

$$a_w |0\rangle_{in} = 0, \tag{34}$$

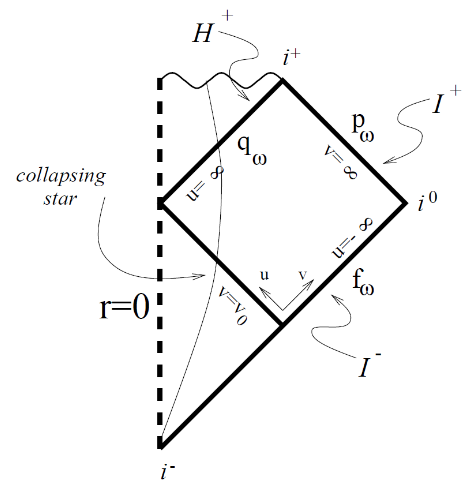

FIG. 3. Penrose diagram for a black hole formed via gravitational collapse: The boundary of the collapsing star is shown. The star interior covers up regions III and IV of the extended black hole spacetime. Spacetime curvature is small inside the star. At some point during collapse, the star falls within its event horizon, and the black hole forms.

for $\forall w > 0$. the label *in* refers to the fact that the boundary conditions on the modes f_w are fixed on I^-.

In order to define a complete basis of particle states at late times, we must find a Cauchy surface at later times. I^+ alone does not form a Cauchy surface (figure 3), so one must choose both I^+ and H^+. On I^+ the *out*-states are taken to be solutions to the wave equation with boundary condition on I^+

$$p_w \to e^{-iwu}. \tag{35}$$

Again this choice of positive frequency late time modes coincides with the usual choice in Minkowski spacetime. To form a complete basis, we must add modes which define particle states on H^+ and its extension through collapsing matter. Here we cannot make a choice based on a flat spacetime limit. Following Hawking's argument one can choose an arbitrary basis and the result of experiments for observers outside the black hole would be the same. to show this, choose any basis q_w that is well behaved on H^+. The density matrix of the system at the early times is simply

$$\rho = |0\rangle_{in\,in}\langle 0| . \tag{36}$$

Expanding ρ in p_w, q_w modes, it is in a tensor product space of the H^+ Fock space and the I^+ Fock space. The expectation value of any operator O^{AF} that only depends on the degrees of freedom in the asymptotically flat region of the spacetime (region I in the usual Kruskal diagrams) may be computed using the reduced density matrix $\rho^{red} = tr_{\{q\}}\rho$ as

$$< O^{AF} >= tr(\rho^{red} O^{AF}). \tag{37}$$

The reduced density matrix is the same for all bases that are related by unitary transformations to the chosen base.

Therefore as promised, $< O^{AF} >$ is independent of the choice of basis on H^+.
Thus we can finally expand the field in the *out*-basis,

$$\phi = \int dw(b_w p_w + b_w^\dagger p_w^* + c_w q_w + c_w^\dagger q_w^*). \quad (38)$$

In order to get a finite result for the number of particles produced in given a frequency interval, per unit time, one must consider wavepackets. Therefore hereafter we use discrete indices to indicate the wavepackets with certain frequencies, i.e. $\{f_i, f_i^*\}$ and $\{p_i, q_i, p_i^*, q_i^*\}$. Then we assume

$$(f_i, f_j) = \delta_{ij}, (p_i, p_j) = (q_i, q_j) = \delta_{ij}, (p_i, q_j) = 0. \quad (39)$$

We therefore have for creation and annihilation operators:

$$a_i = (f_i, \phi), \quad b_i = (p_i, \phi). \quad (40)$$

We can expand

$$p_i = \sum_j (A_{ij} f_j + B_{ij} f_j^*), \quad (41)$$

and so from eq.(11)

$$b_i = (p_i, \phi) = \sum_j (A_{ij}^* a_j - B_{ij}^* a_j^\dagger). \quad (42)$$

The expected number of particles present in the ith *out* mode is then

$${}_{in}\langle 0| b_i^\dagger b_i |0\rangle_{in} = (BB^\dagger)_{ii}. \quad (43)$$

To calculate this we need to determine the Bogolubov coefficients B_{ij}.
We consider for the *out* basis elements p_i so that at I^+ they are wavepackets localized around some u_i and containing only positive frequencies localized around some w_i (figure 4).

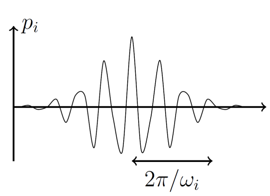

FIG. 4. *Out* basis wavepackets.

Similarly *in* basis elements f_i are chosen to be localized around v on I^+ the same as dependence of p_i on u at I^+.
Consider first the Kruskal spacetime. imagine the wavepacket p_i propagating backwards in time from $I^+ \cup H^+$. Part of the wavepacker would be reflected to give a wavepacket on I^- (an *in* mode) and part would be transmitted to give a wavepacket crossing H^- (an *up* mode). So we can write

$$p_i = p_i^{(1)} + p_i^{(2)}. \quad (44)$$

Let

$$R_i = \sqrt{((p_i^{(1)}, p_i^{(1)}))} \quad T_i = \sqrt{((p_i^{(2)}, p_i^{(2)}))}. \quad (45)$$

Then from the normalization of p_i and the fact that $p_i^{(1)}$ and $p_i^{(2)}$ are othogonal (just like p_i and q_i on $I^+ \cup H^+$), we have

$$R_i^2 + T_i^2 = 1. \quad (46)$$

One can call R_i and T_i reflection and transmission coefficients. Now we can include the collapsing matter in our spacetime. the reflected wavepacket ($p_i^{(1)}$) is scattered of the collapsing matter and this does not experience the time-dependent geometry of the collapsing matter so it just gives a reflected component with unchanged frequency.
On the other hand, the part that would have entered the kruskal spacetime now enters the collapsing matter. Since it has traveled through a time-dependent geometry, the resulting solution will be a mixture of positive and negative modes at I^-. Hense it is $p_i^{(2)}$ that determines B_{ij} (because this mode mixes the modes on I^- resulting creation of particles). Thus we have (as $B_{ij}^{(1)} = 0$)

$$A_{ij} = A_{ij}^{(1)} + A_{ij}^{(2)}, \quad B_{ij} = B_{ij}^{(2)} \quad (47)$$

In order to determine B_{ij} one must determine the behavior of $p_i^{(2)}$ on I^-. on I^+, the wavepacket p_i has oscillations with characteristic frequency near to w_i, modulated by a smooth profile (like a Gaussian function) localized around some u_i. There will be infinitely many oscillations along I^+. When these are propagated backwards in time, there will be infinitely many oscillations between the line $u = u_i$ and the event horizon at $u \to \infty$. See figure 5. Let γ denote a generator of H^+ and extend γ to the past until it intersects I^-. We can define our advanced time coordinate v so that γ intersects I^- at $v = 0$. Our wavepacket will be localized around some value $v_0 < 0$ on I^-, with infinitely many oscillation in $v_0 < v < 0$. Since the field is oscillating so rapidly near γ, we can use the *geometric optic* approximation.
In geometric optics we write the scalar field as $\phi(x) = A(x)e^{i\lambda S(x)}$ (*Eikonal* approximation) and assume $\lambda >> 1$. To leading order in λ the wave equation reduces to $(\nabla S)^2 = 0$, which means that surfaces of constant phase S are null hypersurfaces. The generators of such surfaces are null geodesics.
Next we consider a congruence of null geodesics containing the generators of these surfaces of constant phase, and the also the generators of H^+ (which is the surface S$\to \infty$). Following the approach taken by Reall [7] one can introduce a null vector N^a such that $N.U = -1$ where U^a is the tangent vector to the geodesics and $U.\nabla N^a = 0$. We can decompose a deviation for this congruence into the sum of a part orthogonal to U^a and a term βN^a parallelly transported along geodeics,

$$J^a = \alpha U^a + \beta N^a + \hat{J}^a, \quad U.\hat{J} = N.\hat{J} = 0 \quad (48)$$

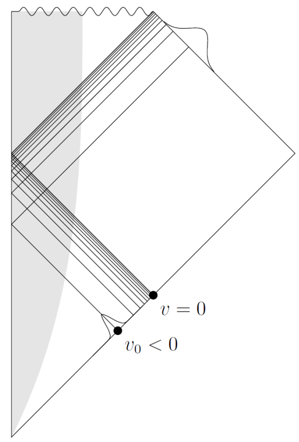

FIG. 5. Surfaces of constant phase accumulate near event horizon and past extension of horizon generators.

where J^a are deviation fields. On H^+, the $J^a - \beta N^a$ part is tangent to H^+ but the βN^a part points off H^+ and hence towards a generator of a surface of a surface of constant S. Choose $\beta = -\epsilon$ where $\epsilon > 0$ is small. Then $-\epsilon N^a$ is a deviation vector from γ to a generator $\gamma^{'}$ of a surface of constant S (figure 6).

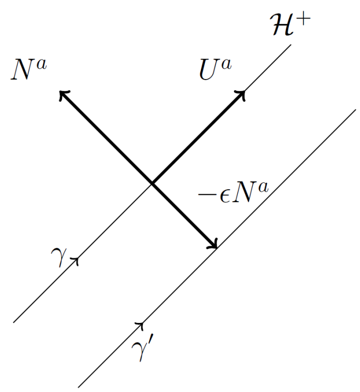

FIG. 6. Schematic of the null congruence mentioned in the calculations.

Spherical symmetry implies that we can choose N^μ such that $N^\theta = N^\phi = 0$. Outside the collapsing matter we know that $\partial/\partial V$ is tangent to the affinely parameterized generators of H^+ (see Townsend [17]), so we can choose $U^a = (\partial/\partial V)^a$ there. Since N^μ is null and not parallel to U^μ we must then have $N^V = 0$. From U.N=-1 we obtain

$$N = C\frac{\partial}{\partial U} \tag{49}$$

for some positive constant C (since g_{UV} is constant on H^+ outside the matter). Hence outside, the collapsing matter, the deviation vector $-\epsilon N^a$ connects γ to a null geodesic $\gamma^{'}$ with

$$U = -C\epsilon. \tag{50}$$

From the definition of U in eq.(15) we have

$$u = -\frac{1}{\kappa}log(-U), \tag{51}$$

where we used the fact that for Schwarzschild geometry $\kappa = 1/(4M)$. Hence, at late time, $\gamma^{'}$ is an outgoing null geodesic with

$$u = -\frac{1}{\kappa}log(C\epsilon). \tag{52}$$

Let $F(u)$ denote the phase of the wavepacket p_i on I^+. Then the phase everywhere along $\gamma^{'}$ must be

$$S = F(-\frac{1}{\kappa}log(C\epsilon)) \tag{53}$$

Now for the early times on I^-, $\gamma, \gamma^{'}$ are ingoing radial null geodesics. In (u, v) coordinates this implies that U^a is a multiple of $\partial/\partial u$. The metric near I^- has the flat form,

$$ds^2 = -dudv + \frac{1}{4}(u-v)^2 d\Omega^2. \tag{54}$$

So spherical symmetry and the fact that N is null and not parallel to U implies

$$N = D^{-1}\frac{\partial}{\partial v} \quad at \quad I^-, \tag{55}$$

for some positive constant D, which implies that $\gamma^{'}$ intersects I^- at

$$v = -D^{-1}\epsilon. \tag{56}$$

Combining with eq.(53), we learn that the phase on I^- is, for small $v < 0$,

$$S = F(-\frac{1}{\kappa}log(-CDv)). \tag{57}$$

Hence on I^- we have

$$p_i^{(2)} \approx \begin{cases} 0 & v > 0 \\ A(v)exp[iF(-\frac{1}{\kappa}log(-CDv))] & \text{small } v < 0 \end{cases} \tag{58}$$

To determine B_{ij} we now have to decompose this function into positive and negative frequency *in* modes on I^-.

So far we have been working with normalizable wavepackets. But now we will assume that p_i contains only the single positive frequency $w_i > 0$ so $F(u) = -w_i u$. This means that p_i is neither normalizable nor localized at late time but it makes the rest of the calculation easier. The result is the same as a more rigorous calculation using wavepackets. Like eq.(35) will use w to label the modes. With the above considerations one can write on I^-:

$$p_i^{(2)} \approx \begin{cases} 0 & v > 0 \\ A_w(v)exp[i\frac{w}{\kappa}log(-CDv)] & \text{small } v < 0 \end{cases}. \tag{59}$$

Similarly we use a basis of *in* modes f_σ such that $f_\sigma = (2\pi N_\sigma)^{-1}e^{-i\sigma v}$ on I^- where N_σ is a normalization constant. Writing $p_w^{(2)}$ in terms of $\{f_\sigma, f_\sigma^*\}$ is therefore just a Fourier transform with respect to v on I^-. Since $p_w^{(2)}$ is squeezed into a small range on v near $v = 0$ (or would be if it were a wavepacket), its Fourier transform will involve mainly high frequency modes, i.e. large σ. For such modes, the Fourier transform is dominated by the region where $p_w^{(2)}$ oscillates most rapidly, i.e. near v=0. So we can use the above expression and approximate the amplitude $A_i(v)$ as a constant. The Fourier transform is therefore

$$\tilde{p}_w^{(2)}(\sigma) = A_w \int_{-\infty}^{0} dv e^{i\sigma v} exp[i\frac{w}{\kappa}log(-CDv)], \quad (60)$$

with inverse

$$\begin{aligned} p_w^{(2)}(v) &= \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} e^{-i\sigma v}\tilde{p}_w^{(2)}(\sigma) \\ &= \int_0^\infty d\sigma N_\sigma \tilde{p}_w^{(2)}(\sigma) f_\sigma(v) + \int_0^\infty d\sigma N_\sigma^* \tilde{p}_w^{(2)}(-\sigma) f_\sigma(v)^* \end{aligned} . \quad (61)$$

The first term picks out the positive frequency components and second term the negative ones. Hence in eq.(41) we have

$$A_{w\sigma}^{(2)} = N_\sigma \tilde{p}_w^{(2)}(\sigma) \quad B_{w\sigma} = N_\sigma^* \tilde{p}_w^{(2)}(-\sigma)\,, w, \sigma > 0 \quad (62)$$

Integral in eq.(60) is not convergent but this is an artifact of working with non-normalizable states. It would converge if we used wavepackets so we will manipulate it as if it converged. With certain analytic extensions in the complex plane one can write:

$$\begin{aligned} \tilde{p}_w^{(2)}(-\sigma) &= -A_w \int_0^\infty dv e^{-i\sigma v} exp[i\frac{w}{\kappa}log(-CDv)] \\ &= -A_w \int_0^\infty dv e^{-i\sigma v} exp[i\frac{w}{\kappa}(log(CDv) + i\pi)] \\ &= -A_w e^{-w\pi/\kappa} \int_{-\infty}^0 dv e^{i\sigma v} exp[i\frac{w}{\kappa}log(-CDv)] \\ &= e^{-w\pi/\kappa}\tilde{p}_w^{(2)}(\sigma) \end{aligned} . \quad (63)$$

Therefore one obtains

$$|B_{w\sigma}| = e^{-w\pi/\kappa}|A_{w\sigma}^{(2)}|. \quad (64)$$

We now return to using wavepackets, for which corresponding result is

$$|B_{ij}| = e^{-w_i\pi/\kappa}|A_{ij}^{(2)}|. \quad (65)$$

Now the normalization of $p^{(2)}$ gives (upon substituting in the decomposition of $p^{(2)}$ in terms of f, f^*)

$$\begin{aligned} T_i^2 = (p_i^{(2)}, p_i^{(2)}) &= \sum_j (|A_{ij}^{(2)}|^2 - |B_{ij}|^2) \\ &= (e^{2w_i\pi/\kappa} - 1)\sum_j |B_{ij}|^2 = (e^{2w_i\pi/\kappa} - 1)(BB^\dagger)_{ii} \end{aligned} . \quad (66)$$

Hence the expected number of late time *out* particles of type i is

$$_{in}\langle 0|\, b_i^\dagger b_i \,|0\rangle_{in} = \frac{\Gamma_i}{(e^{2w_i\pi/\kappa} - 1)}, \quad (67)$$

where $\Gamma_i = T_i^2$. Observers far from the black hole thus see a flux of thermal radiation emitted from the black hole at a temperature proportional to its surface gravity. This is the celebrated Hawking effect, the radiation itself is known as Hawking radiation.

VIII. ACKNOWLEDGMENTS

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