The Information Paradox and The Gravitational Entanglement Entropy*

Kuroush Allameh†

Department of Physics, Sharif University of Technology, Tehran, Iran

Advisor: Ali Naseh

School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

September 28, 2021

Abstract

In this thesis some of the classical and quantum mechanical aspects of the black hole physics including the information paradox are presented. The calculation of the entanglement entropy using the replica trick in the conformal field theories and the gravity is reviewed. Some more recent developments in the context of the information paradox including the islands and replica wormholes are reviewed and some comments on ensemble interpretation of the gravity is presented.

---

*Thesis For B.Sc

†k.allameh@physics.sharif.edu

Contents

1 Introduction

Since the inception of the theory of the general relativity, the black hole solutions have been the center of awe and wonder. A more precise understanding of these objects seems to give us hints about a variety of different aspects of the gravity, from the genesis of the universe to the points of spacetime on which the quantum mechanical effects are non-negligible. Some of the interesting properties of the black holes comes from the classical theory of the general relativity. In fact these properties are the reason why John Wheeler called them black holes in the first place. However the inclusion of quantum effects seem to give even more perplexing results.

In 1975 Hawking in his famous paper showed [1] that inclusion of the quantum mechanical effects would cause the black holes to evaporate and release thermal radiation, with a temperature only depending on a small set of parameters in the black hole. The obvious question that would arise is the fate of the information falling inside the black hole. Indeed no matter how complex an object is if one throws it inside the black hole due to the fact that only a few parameters describe the black hole, after the process of the evaporation the thermal radiation does not seem to return the information in regard to the falling object. This problem is called the black hole information paradox and it has been the center of discussions in quantum gravity for nearly 50 years now, and still there are a lot of puzzles unanswered in this regard.

The concept of information in the quantum mechanics is mainly related to the entanglement entropy, which is defined for a system with the density operator ρ by

$$S = -tr(\rho \ln \rho). \tag{1.1}$$

So one way to study the evolution of the information in the course of the evaporation of the black hole is to try and compute the entanglement entropy in the gravitational setup. Furthermore in the 90s Don Page [2] studied the process of radiation in a unitary setup and provided a theorem which could illustrate the evolution of the entanglement entropy in the course of a unitary radiation. The results by page didn't seem to match the calculations done by Hawking. Thus ever since people have been trying to find a way to explain how one can fit the unitarity and the evaporation of the black holes together.

Various methods can be used to calculate the entanglement entropy but most of them seem to be impossible to apply to the cases of gravity and quantum fields living on it, due to the infinite degrees of freedom in these cases. However a particular method called the replica method, which involves calculating the entanglement Renyi entropies seems to be feasible in these cases. The Renyi entropies are defined by

$$S^{(n)} = \frac{1}{1-n} \ln(tr\rho^n), \tag{1.2}$$

and one proves that these are related to the entanglement entropy by taking a limit

$$\lim_{n\to 1} S^{(n)} = S. \tag{1.3}$$

Specifically for the case of $2D$ conformal field theories (CFT) this method is employed by Cardy and Calabrese [3] giving the right results.

On the other hand Ryu and Takayanagi conjectured [4] that one can calculate the entropy in a CFT using the holography by finding a minimal surface in the bulk homologous to the entangling subsystem in the boundary. Using this prescription they could find the same results of Cardy and Calabrese in using AdS_3/CFT_2.

The use of the replica method in the gravity alone was also studied by various people. In particular, Maldacena and Lewkowycz [5] used this method together with the small inclusion of conical singularities to find the gravitational area law for the entropy. Their method further could explain the conjecture by Ryu and Takayanagi. So one can expect that in a gravitational setup the following formula is right for the entropy:

$$S = \frac{A_{minimal}}{4G}. \tag{1.4}$$

However all of these contributions still could not successfully explain the information paradox and the inconsistency with the Page's theorem was still there. But recently the consideration of new gravitational saddles in the gravitational path integral [6] together with the inclusion of the quantum mechanical corrections [7] to the prescription of the Maldacena and Lewkowycz brought upon new insights and could seemingly give the right behavior for the entropy consistent with Page's theorem. These inclusions give a new region that together with the radiation and the black hole forms a pure state. This new region is called an island and at the late times, the entropy of the radiation subsystem together with the islands will give the right behavior for the entropy, while at early times the islands region does not exist and entropy follows the usual calculations of Hawking. Thus one has:

$$S_{radiation} = min\{S_{islands}, S_{no\ islands}\}. \tag{1.5}$$

The introduction of new saddles to the path integral raises further questions in regard to the nature of the gravity [8] which are in debate nowadays. Furthermore finding the right behavior for the entanglement entropy is not the end of the road. The question of the microstates of the gravity itself are yet to be answered.

The discussions in this introduction so far was to give a brief picture of the concepts that we will discuss in this review. Each of the points above will be explained thoroughly and precisely in the later chapters.

In chapter 2, we will review the some of classical and quantum mechanical aspects of the black holes and state the Page's theorem and the information paradox.
In chapter 3, we review the calculations of the entanglement entropy in the conformal field theories using the replica method.
In chapter 4, we introduce the holographic prescription of the Ryu and Takayanagi and do some calculations using this formula.
In chapter 5, we prove the formula for the gravitational entropy and argue for the Ryu-Takayanagi formula using the prescription of the Maldacena and Lewkowycz.
And finally in chapter 6, we put all of these concepts together and with the use of new gravitational saddles and islands find the right behavior for the entropy of the radiation. we will briefly discuss the implication of the new saddles on the theory of gravity as well.

2 Review of Black Hole Thermodynamics and the Statement of the Information Paradox

In this section we review some classical and quantum mechanical aspects of the black holes and then proceed to state the information problem. Throughout this chapter we assume that we have a manifold furnished with a metric with signature $(-+++)$ as our spacetime.

2.1 Classical Laws of Black Hole Dynamics

This section will be mainly a set of definitions and theorems.

Defintion (Killing Horizon): A null hypersurface $\mathcal{N}$ is called a Killing horizon for a Killing vector field ξ, if ξ is orthogonal to $\mathcal{N}$ on $\mathcal{N}$. For such surfaces one can easily prove [9]:

$$\xi.\nabla\xi^\mu = \kappa\xi^\mu, \quad on\ \mathcal{N}, \tag{2.1}$$

in which

$$\kappa = \left(-\frac{1}{2}(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu)\Big|_{\mathcal{N}} \right)^{\frac{1}{2}}, \tag{2.2}$$

is called the surface gravity.

Theorem (Hawking's Rigidity Theorem): In a stationary, analytic, asymptotically flat vacuum black hole spacetime, horizon is a Killing horizon.

A proof can be found in [10]. This theorem looks a bit abstract but it states that for all of the typical black hole spacetimes that we are familiar with, it is possible to define the surface gravity. In the 70s a series of theorems were proven by Hawking, Carter and Bardeen [11], which characterized the dynamics of black holes. Now we state these, keeping in mind that due to the rigidity theorem it is possible to define the surface gravity for a black hole.

2.1.1 Zeroth Law

This law states that the surface gravity κ is constant on the horizon of a black hole [9].

2.1.2 First Law

This law states that the variations of the energy (mass) of a black hole obeys the following relation [9]:

$$\boxed{G\delta M = \frac{\kappa}{8\pi}\delta A + \Phi\delta Q + \Omega\delta J,} \tag{2.3}$$

where A is the area of the horizon, Φ is the electric potential, Ω is the angular velocity and J is the angular momentum.

2.1.3 Second Law

This law is known as the Hawking's area theorem, and states that in the course of evolution of a black hole, the area of the horizon cannot decrease:

$$\boxed{\delta A \geq 0.} \tag{2.4}$$

In 1973 Bekenstein argued that [12] if black holes are not have entropy, then due to the fact that they are described by a small set of parameters (statement of the no hair theorem), the information of the infalling object will vanish and this will violate the second law of thermodynamics. Thus he stated that by appointing an entropy to the black holes which is proportional to the area of the horizon, one can escape from this violation.
Anyone who is familiar with thermodynamics can easily realize the similarity between the laws of black hole dynamics mentioned in this section, to the laws of thermodynamics. Furthermore Bekenstein's proposal further approves this connection. But this seemed problematic. The laws of black hole dynamics are all theorems proven solely by geometric arguments, and even worse, by definition black holes do not radiate anything to the outside of the horizon in the classical general relativity. If so how can one appoint a temperature to them?
As we shall see the bridge between these laws and the actual thermodynamics is the inclusion of quantum mechanical effects.

2.2 Hawking Radiation

Hawking in his famous paper [1] showed that when general relativity is coupled to quantum field theory (QFT), black holes actually radiate and have a temperature

$$\boxed{T = \hbar \frac{\kappa}{2\pi}.} \tag{2.5}$$

The basis of Hawking's argument in [1], is the idea of semiclassical gravity, in which one can quantize fields quantum mechanically, but keep gravity classical. Hence, one considers quantum field theory in a fixed curved background.

2.2.1 Quantum Fields in Curved Background

In the case of QFTs, it seems reasonable to follow the usual approach in formulating gravitational field theories and make use of the so called "Minimal coupling" principle to write the field expansions in the general covariant form. Indeed this approach works, minus a number of difficulties that are caused due to the loss of the concept of "global families of inertial observers". This issue is more or less the same issue that one would face when trying to make the transition between the special and general theory of relativity.
Perhaps the most important difference to keep in mind is to remember that the concept of particles is frame dependent in curved space, while in the flat space, the Lorentz invariance confirms the frame independence of particles.

for the case of scalar field theory in the curved spacetime, we can demonstrate this issue rather simply. One can choose a complete basis f_w of solutions of the curved scalar wave equation $g^{ab}\nabla_a\nabla_b\phi = 0$. We choose this basis so that the basis functions are delta function orthonormal $(f_w, f_{w'}) = \delta(w - w^{'})$ with respect to the inner product

$$(f, h) = -i \int_{\Sigma} (f\nabla_n h^* - h\nabla_n f^*)\sqrt{|\gamma|}d^3x, \tag{2.6}$$

where the integral is taken over a 3−dimensional Cauchy surface and n is the normal to that surface and γ is the metric induced on it.
The quantum field ϕ can be expanded in the basis as

$$\phi = \int dw(a_w f_w + a_w^{\dagger} f_w^*), \tag{2.7}$$

where a_w and $a_w^{\dagger}$ are operators satisfying

$$[a_{w'}, a_w^{\dagger}] = \delta(w^{'} - w), \quad [a_{w'}, a_w] = [a_{w'}^{\dagger}, a_w^{\dagger}] = 0. \tag{2.8}$$

Finally to fully specify the theory we would define the vacuum states corresponding to a_w,

$$a_w \left|0\right\rangle_a = 0, \tag{2.9}$$

for $\forall w > 0$. One could express the field theory in another arbitrary basis of solutions $\{p_w, p_w^*\}$ equivalently by the expansion

$$\phi = \int dw(b_w p_w + b_w^{\dagger} p_w^*), \tag{2.10}$$

where the analogous commutator expressions to that of a_ws would apply to b_ws. The vacuum state for this basis is similarly given by

$$b_w \left|0\right\rangle_b = 0, \tag{2.11}$$

for $\forall w > 0$. Any two basis of solutions are related to each other by the so called *Bogolubov transformations*

$$\begin{aligned} p_w &= \int dw^{'}(\alpha_{ww'} f_{w'} + \beta_{ww'} f_{w'}^*), \\ f_w &= \int dw^{'}(\alpha_{w'w}^* p_{w'} - \beta_{w'w} p_{w'}^*), \end{aligned} \tag{2.12}$$

where $\alpha_{ww'}$ and $\beta_{ww'}$ are called *Bogolubov coefficients.* Then one can readily check that expansions operators are related by

$$b_w = \int dw^{'}(\alpha_{ww'}^* a_{w'} - \beta_{ww'}^* a_{w'}^{\dagger}). \tag{2.13}$$

With the tools above we can finally calculate the following quantity

$${}_a\left\langle 0\right| (N_w^b) \left|0\right\rangle_a = {}_a\left\langle 0\right| (b_w^{\dagger} b_w) \left|0\right\rangle_a = \int dw^{'} |\beta_{ww'}|^2, \tag{2.14}$$

which states that a vacuum state for a certain basis need not to have zero number of particles for another basis, or in other words the concept of particles is frame dependent.

2.2.2 Particle Creation by Black Holes

Using the black hole geometry and the QFT in curved spacetime introduced above, one can consider the collapsing matter with empty vacuum states in the early times and with the use of 2.14 show that the expected number of late time *out* particles with frequency w_i is

$$\boxed{{}_{in}\langle 0|\, b_i^{\dagger} b_i \,|0\rangle_{in} = \frac{\Gamma_i}{(e^{2w_i\pi/\kappa} - 1)},} \tag{2.15}$$

where Γ_i is a graybody factor, which can be thought of as arising from backscattering of wavepackets off of the gravitational field and into the black hole. This means that if one sits far away in the vacuum and wait long enough for the matter to collapse, eventually she will detect a thermal flux of particles. The complete derivation of this thermal flux can be followed in a clear manner in Hawking's original paper [1].

2.15 is a black body or thermal spectrum, with temperature

$$T = \hbar\frac{\kappa}{2\pi}. \tag{2.16}$$

One fascinating implication is that the classical black hole dynamics theorems and the laws of thermodynamics have more than a formal analogy. According to 2.3, a black hole radiates with temperature $T = \hbar\frac{\kappa}{2\pi}$, and has an entropy

$$S_{bh} = \frac{1}{4}A. \tag{2.17}$$

The entropy above is referred to as the *Bekenstein-Hawking entropy.* Reinstating units we have

$$\boxed{S_{bh} = \frac{c^3 A}{4G\hbar}.} \tag{2.18}$$

The second law of black hole mechanics now states that S_{bh} is non-decreasing classically. Hawking also calculated particle production in quantum fields by charged and rotating black holes. Calculations have also been done for emission of fermions and gravitons and linearized perturbations of the metric. In all of these cases one finds a thermal spectrum,

$$< N_w^{bh} >= \frac{\Gamma_w}{e^{\frac{2\pi(w-\mu)}{\hbar\kappa}} \pm 1}, \tag{2.19}$$

where the +1 corresponds to fermions and -1 to bosons and μ is the chemical potential.
The formula 2.18 is quite general and we shall prove it and it's generalization to non-static cases, when we discuss the work of Maldacena and Lewkowycz.

2.2.3 Black Hole Evaporation

The energy of the Hawking radiation must come from the black hole itself. Hawking's calculation neglects the effect of the radiation on the spacetime geometry. An accurate calculation of this backreaction would involve quantum gravity. However, one can estimate the rate of mass loss by using Stefan's law for the rate of energy loss by a blackbody:

$$\frac{dE}{dt} \approx -\alpha A T^4, \tag{2.20}$$

where α is a dimensionless constant and we approximate Γ_i by treating the black hole as perfectly absorbing sphere of area A (roughly the black hole horizon area) in Minkowski spacetime. Plugging in $E = M$ with $A \propto M^2$ and $T \propto 1/M$ gives $dM/dt \propto -1/M^2$. Hence the black hole evaporates away completely in a time

$$\tau \sim M^3 \sim 10^{71} (\frac{M}{M_\odot})^3 sec. \tag{2.21}$$

This is a very crude calculation but it is expected to be a reasonable approximation at least until the size of the black hole becomes comparable to the Planck mass (1 in our units), when quantum gravity effects are expected to become important. This process of *black hole evaporation* leads to the information paradox, which we discuss in the next section.

2.3 Information Paradox

2.3.1 Page's Theorem

The information paradox is best expressed with the use of the result derived by Page [2]. The Page's theorem relates the size of a subsystem to the amount of randomness of the state in that subsystem if the total state is pure.

Imagine a typical state in the total Hilbert space $\mathcal{H}$, denoted by $|\psi_0\rangle$. Now we define a Haar random state in $\mathcal{H}$ as a state found by applying a Haar random unitary operator on $|\psi_0\rangle$:

$$|\psi(U)\rangle := U |\psi_0\rangle \implies \rho(U) = |\psi(U)\rangle \langle\psi(U)|, \tag{2.22}$$

where $\rho(U)$ is the Haar random density matrix. Next we need a notion of closeness in the Hilbert space. For that we consider L_1 and L_2 norms:

$$L_1: \ ||M||_1 = tr\sqrt{M^\dagger M}, \quad L_2: \ ||M||_2 = \sqrt{tr(M^\dagger M)}, \tag{2.23}$$

where M is an operator in $\mathcal{H}$. Furthermore from linear algebra we have the equivalence of these norms:

$$||M||_2 \leq ||M||_1 \leq \sqrt{N}\, ||M||_2, \tag{2.24}$$

where N is the dimension of the Hilbert space on which operators act. Now we present the theorem in the form that is stated in [13].

Theorem (Page): For any bipartition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}$, one has

$$\boxed{\int dU \left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_1 \leq \sqrt{\frac{|A|^2 - 1}{|A||B| + 1}},} \tag{2.25}$$

where $\rho_A = tr_B(\rho(U))$ is the reduced density matrix on $\mathcal{H}_A$ which is found by tracing out the $\mathcal{H}_B$ degrees of freedom, I_A is the identity operator in $\mathcal{H}_A$ and $|A|, |B|$ are the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The integral is performed with the Haar measure of space of Haar random unitary operators.

Proof: The function $f(x) = x^2$ is convex, so one has the Jensen's inequality $f(\int) \leq \int f$ and can write:

$$\left(\int dU \left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_1\right)^2 \leq \int dU \left(\left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_1\right)^2. \tag{2.26}$$

Also using 2.24:

$$\int dU \left(\left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_1\right)^2 \leq |A| \int dU \left(\left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_2\right)^2 \tag{2.27}$$

Now we write:

$$\begin{aligned}\left(\left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_2\right)^2 &= tr\left(\left(\rho_A(U) - \frac{I_A}{|A|}\right)^\dagger \left(\rho_A(U) - \frac{I_A}{|A|}\right)\right) \\ &= tr\left((\rho_A(U))^2\right) + \frac{1}{|A|} - 2\frac{1}{|A|} = tr\left((\rho_A(U))^2\right) - \frac{1}{|A|}\end{aligned} \tag{2.28}$$

Next step would be to calculate $tr\left((\rho_A(U))^2\right)$. The typical state $|\psi_0\rangle$ is arbitrary, so for simplicity we can choose it to be $|\psi_0\rangle = |k, l\rangle$, where $|k\rangle$ is an orthonormal basis element for $\mathcal{H}_A$ and $|l\rangle$ is an orthonormal basis element for $\mathcal{H}_B$. Now we write $U = \sum_{ij,\beta r} U_{ij,\beta r} |i, j\rangle \langle\beta, r|$ and thus:

$$\begin{aligned}&U |\psi_0\rangle \langle\psi_0| U^\dagger = \sum_{\alpha,\beta}\sum_{q,r} U_{\alpha\beta,kl} U^\dagger_{kl,qr} |\alpha, \beta\rangle \langle q, r| \\ &\implies \rho_A(U) = tr_B(U |\psi_0\rangle \langle\psi_0| U^\dagger) = \sum_i^{|B|}\sum_\alpha^{|A|}\sum_q^{|A|} U_{\alpha i,kl} U^\dagger_{kl,qi} |\alpha\rangle \langle q|,\end{aligned} \tag{2.29}$$

where in the last line above $|\alpha\rangle$ and $|q\rangle$ are orthonormal basis elements for $\mathcal{H}_A$. The readily one can calculate:

$$tr\left((\rho_A(U))^2\right) = \sum_{i,j}^{|B|}\sum_{\alpha,q}^{|A|} U_{\alpha i,kl} U_{qj,k'l'} U^\dagger_{kl,qi} U^\dagger_{k'l',\alpha j}. \tag{2.30}$$

For the Haar measure integrals we have:

$$\begin{aligned}&\int dU = 1 \\ &\int dU\, U_{ij} U_{kl} U^\dagger_{mn} U^\dagger_{op} = \frac{1}{N^2 - 1}(\delta_{ij}\delta_{kp}\delta_{jm}\delta_{lo} + \delta_{ip}\delta_{kn}\delta_{jo}\delta_{lm}) \\ &\qquad - \frac{1}{N(N^2 - 1)}(\delta_{in}\delta_{kp}\delta_{jo}\delta_{lm} + \delta_{ip}\delta_{kn}\delta_{jm}\delta_{lo}).\end{aligned} \tag{2.31}$$

Using 2.30 and 2.31:

$$\int dU tr\Big((\rho_A(U))^2\Big) = \frac{1}{(|A||B|)^2 - 1}(|A||B|^2 + |B||A|^2) \\ - \frac{1}{|A||B|((|A||B|)^2 - 1)}(|A||B|^2 + |B||A|^2) = \frac{|A| + |B|}{|A||B| + 1}. \tag{2.32}$$

So using 2.28 and 2.31:

$$|A| \int dU \left(\left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_2\right)^2 = |A| \int dU \left(tr\Big((\rho_A(U))^2\Big) - \frac{1}{|A|}\right) \\ = \frac{|A|^2 - 1}{|A||B| + 1}. \tag{2.33}$$

So finally with the use of 2.26 and 2.27 and the equation above we have:

$$\int dU \left|\left|\rho_A(U) - \frac{I_A}{|A|}\right|\right|_1 \leq \sqrt{\frac{|A|^2 - 1}{|A||B| + 1}}. \quad Q.E.D. \tag{2.34}$$

This completes the proof for the theorem.

2.3.2 Statement of the Paradox

Now that we have the Page's theorem, we have to interpret it. If we have $|B| \gg |A|$, then $|\rho_A$ would by high probability be close to $I_A/|A|$. This means that the smaller system usually have zero information. The measures of quantum information and entanglement will be discussed in the next chapter when we talk about the entanglement entropy in conformal field theories. For now we just need to realize that entanglement entropy is given by the formula $S_A = -tr(\rho_A \ln \rho_A)$. When the subsystem A is small the entropy would be close to $\ln |A|$ due to the fact that we can approximately say $\rho_A = I_A/|A|$. If we consider a box of radiation that starts to radiate at time

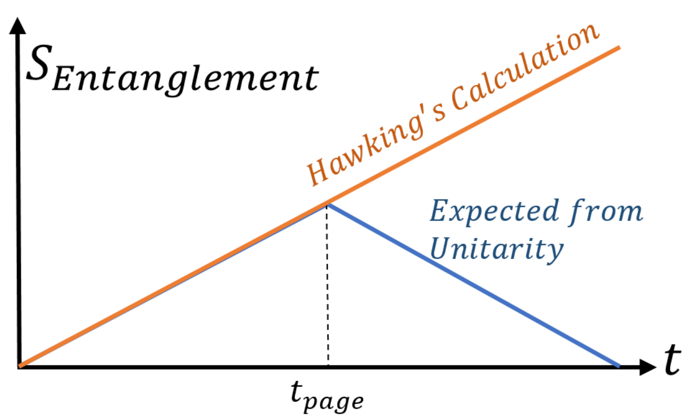


Figure 1: Comparison of the Page curve of a unitary radiation and the Hawking radiation.

$t = 0$, given that the total state of the box and the outside is pure at first the radiation subsystem

in the outside would be small and we can approximate it's entropy by photon gas thermodynamical entropy which is proportional to tT, where T is the temperature and t is the time. After the time t_{page} at which is roughly the half of the total radiation time, the inside subsystem would be smaller and the entropy now can be approxmated by $-tT$.
However if we consider an initial pure state in the case of Hawking's calculations, the radiations subsystem is always in the thermal state and thus the entropy always increases as the time goes by. This is not consistent with the unitary picture that the page's theorem depicts. The plot of entanglement entropy behavior as a funtion of the time is known as the page curve and can be seen in Figure 1.

This behavior of the entanglement in the Hawking radiation implies that if the calculations hold till the end, the purity will not restored. This is a violation of unitarity. In fact, the only possibilities would seem to be that either information is lost during the entire process of formation and evaporation, or the information is restored to the outside world at the very end of the evaporation process, when quantum gravity is at effect. However, we have seen above that after the Page time, the entanglement entropy of the system must decrease, so by the time the black hole has small mass and entropy, the entanglement entropy of the radiation cannot be larger than the black hole's remaining entropy. This comes from the fact that at the Page time black hole is still very big and semiclassical calculations done by Hawking should still work. Thus, even if all information were emitted at the very end of the evaporation process, a law of nature would be violated from the viewpoint of the external observer. The situation is even worse if the information is not emitted at all. Another possibility that was advocated by some authors is that black holes never completely evaporate. Instead they end their lives as stable Planck-mass remnants that contain all the lost information. Obviously such remnants would have to have an enormous, or even infinite entropy. Such objects would be extremely pathological.

With all these being said, recently some arguments have been proposed that give the right behavior of the page curve, and apparently solve the information problem. We shall discuss these in the later chapters.

3 Entanglement in Conformal Field Theories

In this chapter we take a short detour from the gravity and focus on entanglement in conformal field theories. The methods and results in here will be useful later on.

3.1 Entanglement Measures

We first review some general concepts in quantum entanglement before engaging the issue of the entanglement entropy (EE) in conformal field theories (CFT). To talk about the entanglement, one has to consider a bi-partition on the system. We assume that this bi-partition factorizes the Hilbert space into the form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. On a subspace of this Hilbert space one can define a reduced density matrix by integrating the degrees of freedom on the outside. For subsystem A, We define this density matrix to be $\rho_A = tr_B(\rho)$, where the subscript B indicates that trace is taken on the degrees of freedom in the outside subspace, B. Then the usual measure of the entanglement for the subsystem A is the von Neumann entropy or the entanglement entropy defined by

$$\boxed{S_A = -tr(\rho_A \ln \rho_A).} \tag{3.1}$$

There are several reasons to believe why the quantity above is a good measure of entanglement. For example a good property for any entanglement measure is the monotonicity, meaning that with the increase or decrease of the entanglement, the measure must increase or decrease respectively. For further information reader can refer to quantum information literature [14].
For a pure state, $\rho = |\psi\rangle\langle\psi|$, one can easily show that $S_A = S_B = S$. In fact for this state one has the following Schmidt decomposition

$$|\psi\rangle = \sum_a \sqrt{p_a}\, |a,\psi\rangle_A\, |a,\psi\rangle_B\,, \tag{3.2}$$

where $\{|a,\psi\rangle_A\}_a$ are an orthonormal basis for $\mathcal{H}_A$ and $\{|a,\psi\rangle_B\}_a$ are an orthonormal basis for a subspace of $\mathcal{H}_B$ (We assumed that $dim(\mathcal{H}_A) \leq dim(\mathcal{H}_B)$). Now it is evident that $S_A = S_B$, since ρ_A and ρ_B have the same non-zero eigenvalues. Thus the von Neumann entropy really is a good entanglement measure for pure states.

The von Neumann entropy defined above, is a good entanglement measure but it involves calculating the logarithm of an operator, which in high dimensional Hilbert spaces or continuous systems is hard to calculate. One can introduce another entanglement measure called the Rényi entropy. this measure is defined by

$$\boxed{S_A^{(n)} = \frac{1}{1-n}\ln(tr\rho_A^n).} \tag{3.3}$$

This entropy gives the original von Neumann entropy by taking the limit

$$\begin{aligned}\lim_{n\to 1} S_A^{(n)} &= \lim_{n\to 1}\frac{1}{1-n}\ln(tr\rho_A^n) = \frac{\frac{\partial}{\partial n} tr(\rho_A^n)|_{n=1}}{-1} \\ &= -\frac{\partial}{\partial n}\sum_{i=1}^{dim(\mathcal{H}_A)} \lambda_i^n|_{n=1} = -\sum_{i=1}^{dim(\mathcal{H}_A)} \lambda_i \ln(\lambda_i) = -tr(\rho_A \ln(\rho_A)) = S_A,\end{aligned} \tag{3.4}$$

where we assumed the existence of a unique analytic continuation for $tr\rho_A^n$, the L'Hôpital's rule on the second equality, and the basis of eigenvectors of ρ_A with eigenvalues λ_i on the second line. Rényi entropy also has the property $S_A^{(n)} = S_B^{(n)}$ for bi-partitions of pure systems. But why bother with von Neumann entropy when Rényi entropy is easier to compute? The answer lies within the properties that Rényi entropy does not share with von Neumann entropy. One of the most important properties that von Neumann entropy has is the strong subadditivity, $S_{ABC} + S_B \leq S_{AB} + S_{BC}$. Rényi entropy does not have this property. The rest of this chapter is dedicated to the calculation of the entanglement entropy in CFTs and is mainly based on the work of Cardy and Calabrese [3].

3.2 Replica Trick

Both calculations of the von Neumann and the Rényi entropies involve determining a reduced density matrix. This can be a daunting task when dealing with systems with large degree's of freedom or continuous systems like quantum field theories. In order to overcome this obstacle instead of diagonalization one can consult to an alternative method called the ”replica trick”.

In the replica trick, one first evaluates $tr(\rho_A^n)$ or the n-th Rényi entropy, and then given that a uniqune analytic continuation exists, differentiates it with respect to n. Then one takes the limit $n \to 1$. More explicitly,

$$\lim_{n\to 1} \frac{tr(\rho_A^n) - 1}{1-n} = -\frac{\partial}{\partial n} tr(\rho_A^n)|_{n=1} = \lim_{n\to 1} S_A^{(n)} = S_A.$$

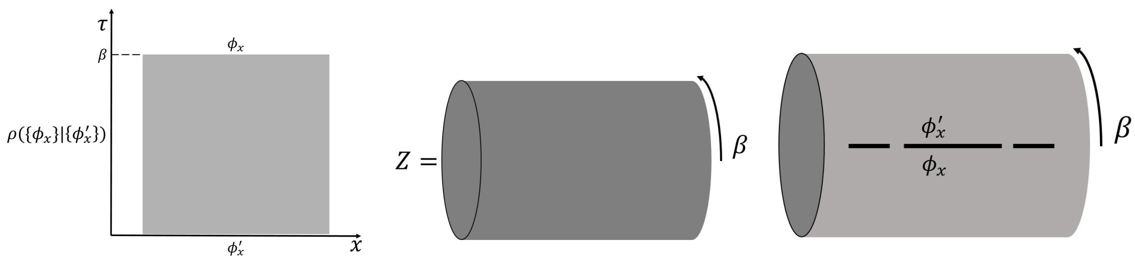
Playing with the expression above one can also show,

$$\begin{aligned} S_A = -\frac{\partial}{\partial n} tr(\rho_A^n)|_{n=1} &= -\sum_{i=1}^{dim(A)} \lambda_i \ln(\lambda_i) \\ &= -\frac{\sum_{i=1}^{dim(A)} \lambda_i \ln(\lambda_i)}{\sum_{i=1}^{dim(A)} \lambda_i} = -\frac{\partial}{\partial n} \ln(tr(\rho_A^n))|_{n=1}, \end{aligned}$$

where in the second line, We used the fact that $\sum_{i=1} \lambda_i = 1$. So replica trick is summarized in the following relations:

$$\boxed{S_A = \lim_{n\to 1} S_A^{(n)} = -\frac{\partial}{\partial n} \ln(tr(\rho_A^n))|_{n=1} = -\frac{\partial}{\partial n} tr(\rho_A^n)|_{n=1}.} \tag{3.5}$$

Therefore what we really have to do is to calculate $tr(\rho_A^n)$ in our system. We will compute $tr(\rho_A^n)$ for positive integral n, then given an analytic continuation exists we find the expression for a general complex value. Existence of this analytic continuation is often guaranteed by the Carslon's theorem [15]. For a further discussion on the uniqueness of analytic continuation, reader can consult to [16] One thing that one has to realize is that the calculations that we did in order to take the limit of the Rényi entropies and replica trick are all done with the assumption of finiteness of dimension of the Hilbert space. But later we apply these results to the quantum field theories and so long as nothing inconsistent happens we take them for granted.



(a) (b) (c)

Figure 2: (a) Path integral representation of $\rho(\{\phi_x\}|\{\phi'_{x'}\})$. (b) The partition function Z is obtained by sewing together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β. (c) The reduced density matrix ρ_A is obtained by sewing together only those points x which are not in A.

3.3 Path Integrals and Riemann Surfaces

First we consider a lattice quantum system with a discrete lattice coordinate, x in one dimension and a continuous time dimension. domain of x can be finite, semi-infinite or infinite. at a given time each point on the lattice is described by a Hilbert space $\mathcal{H}_x$. Thus the full Hilbert space of the system is $\mathcal{H} = \bigotimes_x \mathcal{H}_x$. At each point x one assumes the existence of complete set of commuting observables (CSCO), which can be denoted by $\{\hat{\phi}_x\}$. Then the basis of joint eigenvectors of these observables would fully characterize the Hilbert space at each point. Using the basis of joint eigenstates of these observables we have a basis for the full Hilbert space, $\{\otimes_x |\{\phi_x\}\rangle \coloneqq |\Pi_x\{\phi_x\}\rangle\}$. One can consider an example of two operators at each point $\{\hat{O}_x, \hat{O'}_x\}$ with eigenvalues, O^1_x, O^2_x and O'^1_x, O'^2_x respectively. Then the basis at each point is,

$$\{|\{\phi_x\}\rangle\} = \{|(O^1_x, O'^1_x)\rangle\, , |(O^1_x, O'^2_x)\rangle\, , |(O^2_x, O'^1_x)\rangle\, , |(O^2_x, O'^2_x)\rangle\}.$$

And the basis of the whole space at a given time is then given by $\{|\Pi_x(O^i_x, O'^j_x)\rangle\}$.

Time development of our quantum field theory is governed by a Hamiltonian H, so if we consider the system to be at the inverse temperature β, the elements of density matrix ρ of the system are given by

$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) \coloneqq \langle \prod_x \{\phi_x\}|\, \rho\, |\prod_{x'} \{\phi'_{x'}\}\rangle = Z(\beta)^{-1} \langle \prod_x \{\phi_x\}|\, e^{-\beta H}\, |\prod_{x'} \{\phi'_{x'}\}\rangle\, , \tag{3.6}$$

where $Z(\beta) = tr(e^{-\beta H})$ is the partition function. One can rewrite the formula above in terms of a path integral on the imaginary time interval $(0, \beta)$:

$$\rho(\{\phi_x\}|\{\phi'_{x'}\}) = Z^{-1} \int [d\phi(y.\tau)] \prod_{x'} \delta(\phi(y,0) - \phi'_{x'}) \prod_x \delta(\phi(y,\beta) - \phi_x) e^{-S_E}, \tag{3.7}$$

where the euclidean action is $S_E = \int_0^\beta \mathcal{L} d\tau$, with $\mathcal{L}$ being the euclidean Lagrangian (Figure 2.a). If the exact mathematical meaning the delta functions in the eq.3.7 is a bit ambiguous, one can only keep in mind that the path integral should be evaluated in a way that the initial and final configurations of the field are suitably fixed. The partition function Z is found by setting

$\{\phi_x\} = \{\phi'_{x'}\}$ and integrating over these variables. This means sewing together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β (Figure 2.b).

Now we can consider A to be the subsystem consisting of the points x in disjoint intervals $(u_1, v_1), ..., (u_N, v_N)$. We can modify the path integral above in order to calculate the reduced density matrix ρ_A by integrating out the fields in points x which are not in A. This means sewing together the cylinder at $x \notin A$ and leaving open cuts at $x \in A$, along the line $\tau = 0$ as can be seen from the (Figure 2.c).
For the replica trick as mentioned before, the relevant quantity to be calculated is $tr(\rho_A^n)$. We can calculate this quantity for any positive integer n, by making n copies of the above, labeled by an integer j with $1 \leq j \leq n$, and sewing them together cyclically along the cuts so that $\phi_j(x, \tau = \beta^-) = \phi_{j+1}(x, \tau = 0^+)$ and $\phi_n(x, \tau = \beta^-) = \phi_1(x, \tau = 0^+)$ for all $x \in A$. This defines an n-sheeted structure depicted for $n = 3$ and $N = 1$ (one interval) in (Figure 3). We thus find that

$$tr(\rho_A^n) = \frac{Z_n(A)}{Z^n}, \tag{3.8}$$

where $Z_n(A)$ is the path integral or partition function on the n-sheeted structure with relevant set of cuts, A and appropriate constraints on the fields. Finally given that there exists a unique analytic continuation of the expression above for all $n \in \mathbb{C}$ with $Re(n) > 1$, we can find the entanglement entropy using the replica trick:

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}. \tag{3.9}$$

Now we can talk about the continuum limit in space coordinates. this can be done by taking the limit in which the lattice spacing goes to zero. Then we have $x \in \mathbb{R}$, and the path integral is over fields $\phi(x, \tau)$ on an $n - sheeted\ Riemann\ surface$, with branch points at u_j and v_j. We denote such a surface with $\mathcal{R}_{n,N}$, where n is the number of sheets and N is the number of cuts on each sheet. The branch points are points of non-zero curvature which must be treated with care. With consideration of these points the uniformization theorem in the complex analysis allows us to find a holomorphy (a conformal transformation) between $\mathcal{R}_{n,N}$ and the $\mathbb{C}\backslash\Lambda$, where Λ represents a set of points in $\mathbb{C}$ which correspond to the branch points on the Riemann surfaces. In the next sections when we deal with calculations for curtain cases we will see what these holomorphies are for each case.

3.4 Twist fields

The result of the last section can be formally written in the form of

$$Z_n(A) = Z_{\mathcal{R}_{n,N}} = \int [d\phi]_{\mathcal{R}_{n,N}} exp\Big[- \int_{\mathcal{R}_{n,N}} dx d\tau \mathcal{L}[\phi](x, \tau) \Big]. \tag{3.10}$$

In the relation above we are dealing with a surface with non-zero curvature only at a finite number of points u_j and v_j. Due to locality, we expect to be able to express the partition function as an object that is calculated from a model on $\mathbb{C}$, where appropriate boundary conditions are implemented around the branch points. We form a model by n independent copies of our original field, where

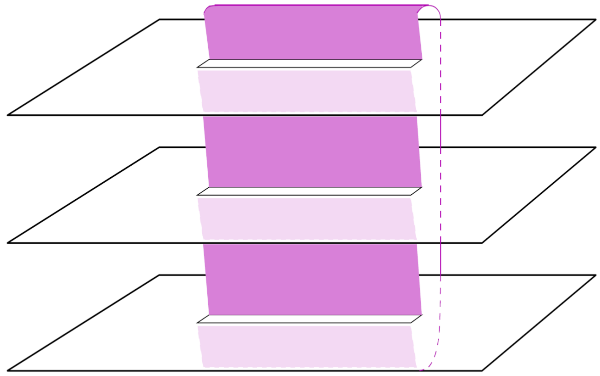


Figure 3: n-sheeted structure for the replica trick.

n is the number of the Riemann sheets necessary to describe the Riemann surface by coordinates on $\mathbb{C}$, or basically it is the number of sheets in the original Riemann surface. For a single interval $[u_1, v_1]$ this means that we basically write

$$Z_{\mathcal{R}_{n,1}} = \int_{C(u_1,0),C(v_1,0)} [d\phi_1...d\phi_n] exp\Big[- \int_{\mathbb{C}} dx d\tau (\mathcal{L}[\phi_1](x,\tau) + ... + \mathcal{L}[\phi_n](x,\tau))\Big], \tag{3.11}$$

where we just duplicated the theory n-times with no interaction and wrote the action as the sum of the action of the each field. The important thing is that now we work on the complex plane with n identical fields with appropriate boundary conditions, rather than a n-sheeted Riemann surface with a single field. This procedure is a recurring theme throughout our discussions and it is basically taking the $\mathbb{Z}_n$ orbifold of the geometry. We will further explain it in different occasions later on. The restriction $\int_{C(u_1,0),C(v_1,0)}$ means:

$$\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-), \quad x \in [u_1, v_1], \quad i = 1, ..., n \tag{3.12}$$

where we have $n + i \equiv i$. The Lagrangian density of the multiple model as explained above is the sum of individual non-interacting models:

$$\mathcal{L}^{(n)}[\phi_1, ..., \phi_n](x, \tau) = \sum_{i=1}^{n} \mathcal{L}[\phi_i](x, \tau). \tag{3.13}$$

Eq.3.11 defines a set of local fields which are special cases of "twist fields". Twist fields exist in a QFT model whenever there is a global internal symmetry σ (a symmetry that acts the same way everywhere in space, and that does not change the positions of fields): $\int_{\mathbb{C}} dx d\tau \mathcal{L}[\sigma\phi](x, \tau) = \int_{\mathbb{C}} dx d\tau \mathcal{L}[\phi](x, \tau)$. When we call these objects "fields", we are referring to a field in the most general QFT sense, meaning they are objects for which correlation functions can be evaluated, and which depend on points in spacetime. We will denote them by $\mathcal{T}_\sigma(x, \tau)$ and their correlation functions are defined through the path integral:

$$\boxed{\langle \mathcal{T}_\sigma(a, b)...\rangle_{\mathcal{L},\mathbb{C}} \propto \int_{C_\sigma(a,b)} [d\phi]_{\mathbb{C}} exp\Big[- \int_{\mathbb{C}} dx d\tau \mathcal{L}[\phi](x, \tau)\Big]...,} \tag{3.14}$$

where ... represents the insertion of other local fields at different points and the path integral conditions are

$$C_\sigma(a,b) \quad : \quad \phi(x,b^+) = \sigma\phi(x,b^-), \quad x \in [a,\infty). \tag{3.15}$$

Let us return to our model of $\mathcal{L}^{(n)}$. It is obvious that this n-fold, non-mutually interacting copies have a permutation symmetry. Any permutation of the fields is equivalent to any other. This symmetry defines a special kind of twist fields called the $branch-point\ twist\ fields$. These correspond to two opposite cyclic permutation symmetries $i \to i+1$ and $i+1 \to i (i = 1, ..., n,\ n+1 \equiv 1)$. We will denote them simply by $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$, respectively

$$\begin{aligned} \mathcal{T}_n \equiv \mathcal{T}_\sigma, \quad \sigma \quad &: i \to i+1 \ mod\, n, \\ \tilde{\mathcal{T}}_n \equiv \mathcal{T}_{\sigma^{-1}}, \quad \sigma^{-1} \quad &: i+1 \to i \ mod\, n. \end{aligned} \tag{3.16}$$

Now using the definition of correlators of these fields as defined in 3.14 we have:

$$\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} \propto \int_{C(u_1,0),C(v_1,0)} [d\phi_1 ... d\phi_n] exp\Big[- \int_{\mathbb{C}} dx d\tau \mathcal{L}^{(n)}[\phi_1,...,\phi_n](x,\tau)\Big]. \tag{3.17}$$

Two conditions $C(u_1,0), C(v_1,0)$ together give:

$$\begin{aligned} &\phi_i(x,0^+) = \phi_{i+1}(x,0^-), \quad x \in [u_1,\infty), \quad i = 1,...,n \\ &\phi_{i+1}(x,0^+) = \phi_i(x,0^-), \quad x \in [v_1,\infty), \quad i = 1,...,n \quad \Longrightarrow \\ &\qquad \phi_i(x,0^+) = \phi_{i+1}(x,0^-), \quad x \in [u_1,v_1], \quad i = 1,...,n \end{aligned} \tag{3.18}$$

which is precisely the condition 3.12 and as seen above the two-point correlation function of these two fields is proportional to 3.11. More generally one can calculate the correlation functions of the original model with $\mathcal{L}$ on $\mathcal{R}_{n,1}$ with consideration of the normalization:

$$\boxed{\langle O(x,\tau; sheet\ \ i)...\rangle_{\mathcal{L},\mathcal{R}_{n,1}} = \frac{\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0) O_i(x,\tau)...\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u_1,0)\tilde{\mathcal{T}}_n(v_1,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}},} \tag{3.19}$$

where O_i is the field in the model $\mathcal{L}^{(n)}$ coming from the i^{th} copy of $\mathcal{L}$. The same expression with the products of more twist and anti-twist fields holds in the case of $\mathcal{R}_{n,N}$.
Consider the model $\mathcal{L}$ to be a CFT. Then also $\mathcal{L}^{(n)}$ is a CFT. There are n fields $T_j(w)$, in $\mathcal{L}^{(n)}$ that correspond to the stress-energy tensors of the n copies of $\mathcal{L}$, and in particular the sum $T^{(n)}(w) = \sum_{j=1}^n T_j(w)$ is the stress-energy tensor of $\mathcal{L}^{(n)}$. The central charge of $\mathcal{L}^{(n)}$ is nc, if c is that of $\mathcal{L}$.

3.5 Single Intervals

Let us consider the case for $N = 1$ meaning that there is only one interval. We also assume that there are no boundaries. Interval is denoted by $[u,v]$ with the length $l = |u-v|$ in an infinitely long $1D$ quantum system, with zero temperature. The complex coordinates are denoted by $w = x + i\tau$ and $\bar{w} = x - i\tau$. The conformal transformation $w \to \zeta = \frac{w-u}{w-v}$ then maps the branching interval to $(0,\infty)$. Then one uses the mapping $\zeta \to \zeta^{1/n}$ which transforms complex plane to a pizza slice shaped region. This transformation gives us a uniformization that maps $\mathcal{R}_{n,1}$ to $\mathbb{C}$ minus some points of branching. This is exactly what we expected from the uniformization theorem in complex analysis as we mentioned earlier. From $2D$ conformal field theory, one knows that the energy

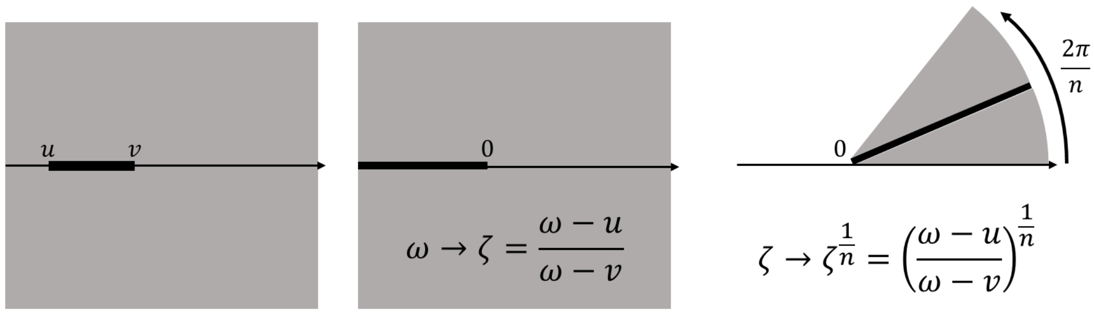


Figure 4: Series of holomorphies, used for the uniformization of the Riemann surface.

momentum tensor has only two non-vanishing components, one being a holomorphic function $T(w)$ and the other an anti-holomorphic function $\bar{T}(\bar{w})$. One also knows that $T(w)$ transforms in the following way:

$$T'(w) = \left(\frac{\partial f}{\partial w}\right)^2 T(f(w)) + \frac{c}{12}\{f(w), w\}, \tag{3.20}$$

where $f(w)$ is a conformal transformation, and $\{f(w), w\} = (f'''f' - \frac{3}{2}f''^2)/f'^2$ is the Schwarzian derivative. We can always redefine the energy momentum tensor as $\tilde{T} = T - \langle T\rangle_{\mathbb{C}}$, so that $\langle \tilde{T}\rangle_{\mathbb{C}} = 0$. So one point functions of the stress tensor can be made to vanish. We thus find

$$\langle T(w)\rangle_{\mathcal{R}_{n,1}} = \frac{c}{12}\{f(w), w\} = \frac{c(1-n^{-2})}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}. \tag{3.21}$$

This expression is related to the twist fields by 3.19,

$$\langle T(w)\rangle_{\mathcal{R}_{n,1}} = \frac{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)T_j(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}, \forall j. \tag{3.22}$$

Then for obtaining the complete correlation function involving all copies we multiply the expression above by n (as we mentioned earlier central charge of $\mathcal{L}^{(n)}$ is nc):

$$\begin{aligned} &\frac{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}} = \frac{c(n^2-1)}{24n}\frac{(v-u)^2}{(w-u)^2(w-v)^2} \\ \Longrightarrow\ &\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} = \frac{c(n^2-1)}{24n}\frac{(v-u)^2}{(w-u)^2(w-v)^2}\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}. \end{aligned} \tag{3.23}$$

We also have the conformal Ward identity:

$$\begin{aligned} &\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)T^{(n)}(w)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} \\ &= \Big(\frac{1}{w-u}\frac{\partial}{\partial u} + \frac{h_{\mathcal{T}_n}}{(w-u)^2} + \frac{1}{w-v}\frac{\partial}{\partial v} + \frac{h_{\tilde{\mathcal{T}}_n}}{(w-v)^2}\Big)\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}. \end{aligned} \tag{3.24}$$

The expressions above with the help of two-point function expression for primary fields in CFT, $\langle \mathcal{T}_n(u,0)\tilde{\mathcal{T}}_n(v,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} = (v-u)^{-2h_{\mathcal{T}_n}-2h_{\tilde{\mathcal{T}}_n}}$, helps one to identify the conformal weights,

$$h_{\mathcal{T}_n} = h_{\tilde{\mathcal{T}}_n} = \frac{c}{24}\left(n-\frac{1}{n}\right). \tag{3.25}$$

Now since we have $tr(\rho_A^n) \propto Z_n(A)/Z^n$, therefore we conclude that this object behaves under scale and conformal transformations identically to the two-point function of primary fields with conformal weights $h_{\mathcal{T}_n}$ and $h_{\tilde{\mathcal{T}}_n}$. It means:

$$tr(\rho_A^n) = c_n\left(\frac{v-u}{a}\right)^{-\frac{c}{6}(n-\frac{1}{n})}, \tag{3.26}$$

where a in the expression above is a constant to make the expression dimensionless. Constants c_n cannot be determined by this method, but $c_1 = 1$. The analytic continuation of this function to $Re(n) > 0$ comes from the Carlson's theorem, as we mentioned earlier. Now we can readily calculate:

$$\begin{aligned} S_A^{(n)} &= \frac{1}{1-n}\ln(tr(\rho_A^n)) = -\frac{c}{6}\frac{n^2-1}{n(1-n)}\ln(\frac{v-u}{a}) + \frac{1}{1-n}\ln c_n \\ &= \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\frac{l}{a} + c_n', \end{aligned} \tag{3.27}$$

where we set $c_n' = \frac{lnc_n}{1-n}$ and $v-u=l$. Now we finally calculate the von Neumann entropy:

$$\boxed{S_A = \lim_{n\to 1} S_A^{(n)} = \frac{c}{3}\ln\frac{l}{a} + c_1'.} \tag{3.28}$$

One must notice that since $c_1 = 1$, using the L'Hôpital's rule $c_1' = -\frac{\partial}{\partial n} l\ln c_n|_{n=1} = -\frac{\frac{\partial}{\partial n}c_n}{c_n}|_{n=1} = -\frac{\partial}{\partial n}c_n|_{n=1}$.

3.6 Generalizations

Now that we have an expression for $tr(\rho_A^n)$, due to the fact that it is proportional to the two-point function of two primary fields, $\mathcal{T}_n$ and $\tilde{\mathcal{T}}_n$ we can make some generalizations using the conformal transformations, $w \to f(w)$, with the formula

$$\langle \mathcal{T}_n(w_1,\bar{w}_1)\tilde{\mathcal{T}}_n(w_2,\bar{w}_2)\rangle = \prod_{j=1}^{2}\big(\frac{\partial f}{\partial w}|_{w=w_j}\big)^{h_j}\big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=\bar{w}_j}\big)^{\bar{h}_j}\langle \mathcal{T}_n(f(w_1),\bar{f}(w_1))\tilde{\mathcal{T}}_n(f(w_2),\bar{f}(w_2))\rangle. \tag{3.29}$$

3.6.1 Finite Temperature or Finite Size

First we can consider the case for finite temperature, i.e. $\beta = finite$. Here one needs to map a cylinder with circumference β to the punctured complex plane. This can be done with the mapping

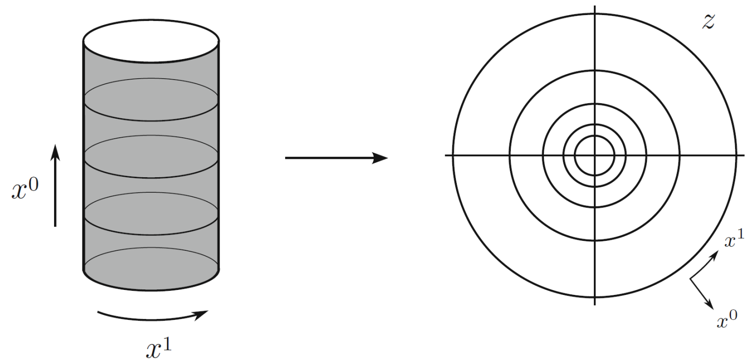


Figure 5: Mapping $z = f(\omega) = e^{\frac{2\pi}{\beta}w}$ from cylinder to the complex plane.

$z = f(w) = e^{\frac{2\pi}{\beta}w}$, where $f(w)$ are now coordinates in the punctured complex plane which we know the two point function for (Figure 5). This mapping is conformal since $dzd\bar{z} = (\frac{2\pi}{\beta})^2 e^{2Re(w)} dwd\bar{w}$. Using 3.29 above, we can write:

$$\begin{aligned}
tr(\rho_A^n)_{Cylinder} &= \big(\frac{\partial f}{\partial w}|_{w=u}\big)^{h_n}\big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=u}\big)^{h_n}\big(\frac{\partial f}{\partial w}|_{w=v}\big)^{h_n}\big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=v}\big)^{h_n}\langle \mathcal{T}_n(f(u),0)\tilde{\mathcal{T}}_n(f(v),0)\rangle_{\mathbb{C}} \\
&= \big(\frac{\partial}{\partial w}(e^{\frac{2\pi}{\beta}w})|_{w=u}\big)^{h_n}\big(\frac{\partial}{\partial \bar{w}}(e^{\frac{2\pi}{\beta}\bar{w}})|_{\bar{w}=u}\big)^{h_n}\big(\frac{\partial}{\partial w}(e^{\frac{2\pi}{\beta}w})|_{w=v}\big)^{h_n}\big(\frac{\partial}{\partial \bar{w}}(e^{\frac{2\pi}{\beta}\bar{w}})|_{\bar{w}=v}\big)^{h_n} \\
&\langle \mathcal{T}_n(f(u),0)\tilde{\mathcal{T}}_n(f(v),0)\rangle_{\mathbb{C}} \\
&= \big(e^{\frac{4\pi v}{\beta}}e^{\frac{4\pi u}{\beta}}(\frac{2\pi}{\beta})^4\big)^{\frac{c}{24}(n-1/n)}\frac{c_n}{((exp(\frac{2\pi v}{\beta}) - exp(\frac{2\pi u}{\beta}))a^{-1})^{(\frac{c}{6}(n-1/n))}} \\
&= c_n\big(\frac{\beta}{2\pi a}(e^{\frac{\pi}{\beta}(v-u)} - e^{\frac{\pi}{\beta}(u-v)})\big)^{-\frac{c}{6}(n-1/n)} \\
&= c_n\big(\frac{\beta}{\pi a}\sinh(\frac{\pi}{\beta}(v-u))\big)^{-\frac{c}{6}(n-1/n)},
\end{aligned} \tag{3.30}$$

where we set $h_n = h_{\tilde{\mathcal{T}}_n} = h_{\mathcal{T}_n}$. So we have:

$$\begin{aligned}
S^{(n)}_{A_{Cylinder}} &= \frac{1}{1-n}\ln(tr(\rho_A^n)_{Cylinder}) = -\frac{c}{6}\frac{n^2-1}{n(1-n)}\ln\big(\frac{\beta}{a\pi}\sinh(\frac{\pi}{\beta}(v-u))\big) + c'_n \\
&= \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left(\frac{\beta}{a\pi}\sinh(\frac{\pi}{\beta}(v-u))\right) + c'_n \\
\implies & \boxed{S_{A_{Cylinder}} = \lim_{n\to 1} S^{(n)}_{A_{Cylinder}} = \frac{c}{3}\ln\left(\frac{\beta}{\pi a}\sinh(\frac{\pi l}{\beta})\right) + c'_1.}
\end{aligned} \tag{3.31}$$

The limiting cases for this expression can be easily obtained using the limits $l \ll \beta$ and $l \gg \beta$, which in turn give $\frac{c}{3}\ln\frac{l}{a} + c'_1$ and $\frac{\pi c}{3\beta}l + c'_1$. The first limit is the zero temperature limit which we calculated earlier, and the other one is the high temperature limit which gives rise to pure thermal entropy of a $1D$ photon gas.

Alternatively one can orient the branch cut perpendicular to the axis of the cylinder. This corresponds to the entropy of a subsystem of length l in a finite $1D$ system of length L, with periodic boundary conditions. a mapping that takes one from the finite system to the complex plane is $f(w) = \tan(\frac{\pi}{L}w)$, where $f(w)$ are now points in complex plane which we know the two point function for them. We can double check that this mapping is suitable, by pointing out that at $w \to \pm\frac{L}{2}$ one has $f(w) \to \pm\infty$ which is consistent with mapping a two ends of a finite interval to the positive and negative infinities in the complex plane. Now let us redo the calculations reminiscent of those of eq.3.30.

$$\begin{aligned} tr(\rho_A^n)_{finite-length} &= \big(\frac{\partial f}{\partial w}|_{w=u}\big)^{h_n}\big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=u}\big)^{h_n}\big(\frac{\partial f}{\partial w}|_{w=v}\big)^{h_n}\big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=v}\big)^{h_n}\langle\mathcal{T}_n(f(u),0)\tilde{\mathcal{T}}_n(f(v),0)\rangle_{\mathbb{C}} \\ &= \big((\frac{\pi}{L})^4\frac{1}{\cos^4(\frac{\pi}{L}u)}\frac{1}{\cos^4(\frac{\pi}{L}v)}\big)^{\frac{c}{24}(n-1/n)}\frac{c_n}{((\tan(\frac{\pi}{L}v)-\tan(\frac{\pi}{L}u))a^{-1})^{(\frac{c}{6}(n-1/n))}} \\ &= c_n\big(\frac{L}{\pi a}(\sin(\frac{\pi}{L}v)\cos(\frac{\pi}{L}u)-\sin(\frac{\pi}{L}u)\cos(\frac{\pi}{L}v))\big)^{-\frac{c}{6}(n-1/n)} \\ &= c_n\big(\frac{L}{\pi a}\sin(\frac{\pi}{L}(v-u))\big)^{-\frac{c}{6}(n-1/n)}. \end{aligned} \tag{3.32}$$

So we have:

$$\begin{aligned} S^{(n)}_{A_{finite-length}} &= \frac{1}{1-n}\ln(tr(\rho_A^n)_{Cylinder}) = -\frac{c}{6}\frac{n^2-1}{n(1-n)}\ln\big(\frac{L}{a\pi}\sin(\frac{\pi}{L}(v-u))\big) + c'_n \\ &= \frac{c}{6}\left(1+\frac{1}{n}\right)\ln\left(\frac{L}{a\pi}\sin(\frac{\pi}{L}(v-u))\right) + c'_n \\ \Longrightarrow &\ \boxed{S_{A_{finite-length}} = \lim_{n\to 1} S^{(n)}_{A_{finite-length}} = \frac{c}{3}\ln\left(\frac{L}{\pi a}\sin(\frac{\pi l}{L})\right) + c'_1.} \end{aligned} \tag{3.33}$$

3.6.2 Several Intervals

The general case for a set of disjoint intervals is rather complicated. In this case the Riemann surface is $\mathcal{R}_{n,N}$ meaning that one has:

$$A = \{w|Im(w) = 0, Rew \in [u_1, v_1] \cup [u_2, v_2] \cup ... \cup [u_N, v_N]\}. \tag{3.34}$$

For this case the resulting replicated geometry $\mathcal{R}_{n,N}$ has genus $(n-1)(N-1)$. So unlike the cases above, there won't be a uniformization to the complex plane. The result in this case is generally dependent on the field content of the theory and is not universal. We mention the result for c free Dirac fermions here [17], because we will be needing them later on. If we have two intervals

$$[u_1, v_1] \cup [u_2, v_2]. \tag{3.35}$$

Then for the case of zero temperature and infinitely long system one has:

$$S_{fermions} = \frac{c}{3}\ln\left(\frac{|v_1-u_1||u_2-v_1||v_2-u_2||v_2-u_1|}{|u_2-u_1||v_2-v_1|}\right). \tag{3.36}$$

Using the conformal transformations like the cases above, one can easily relate this result to any case with a metric $ds^2 = \Omega^{-2}ds^2_{flat}$ as we shall see later.

4 Holographic Prescription

After discussing a bit about the concept of entanglement entropy in CFTs it is time to give a prescription for calculating the entanglement entropy of region using the AdS/CFT correspondence, first introduced by Ryu and Takayanagi (RT) [4]. This holographic prescription will reproduce the results of the last chapter and also inspire ideas that will be useful for us later.

4.1 Anti-de Sitter (AdS) spacetime

Before stating the RT prescription it is good to review some of aspects of the AdS spacetimes. Anti-de Sitter space, often abbreviated AdS, is an exact solution of Einstein's field equations of general relativity characterized by a constant negative spacetime curvature and a vanishing stress-energy tensor (vacuum solution). The $d+1$ dimensional AdS (or more precisely "univesal covering space of the anti-de Sitter" space denoted by $CAdS$) spacetime is a manifold homeomorphic to $\mathbb{R}^{1,d}$ with a semi-Riemannian metric:

$$\boxed{ds^2 = R_{AdS}^2\big(- \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2\big),} \tag{4.1}$$

where $\rho \geq 0$, $\tau \in \mathbb{R}$ is the proper time coordinate and Ω_{d-1}^2 is the metric of the S^{d-1} sphere. The quantity R_{AdS} is called the AdS radius. This space is maximally symmetric, meaning it has $\frac{D(D+1)}{2}$ Killing vectors, where $D = d+1$ and consequently it has a constant negative curvature. In order to understand the intuition of the definitions and names above we can follow a usual way of looking at this spacetime by looking at it as a submanifold embedded in the flat, $d+2$ dimensional semi-Riemannian manifold $\mathbb{R}^{2,d}$ with the metric $\eta_{\mu\nu} = diag(-1,-1,1,...,1)$. If we denote the two timelike coordinates x^0 and x^{d+1} then we have for the embedding space:

$$ds^2 = -dx^0 - d(x^{d+1})^2 + \sum_{i=1}^{d} d(x^i)^2. \tag{4.2}$$

Then the AdS spacetime is defined as a submanifold satisfying:

$$-(x^0)^2 - (x^{d+1})^2 + \sum_{i}^{d} (x^i)^2 = -R_{AdS}^2. \tag{4.3}$$

The embedding space is a $d+2$-dimensional flat space, thus it possesses the maximal number of symmetries allowed in $d+2$ dimensions, namely $(d+2)(d+3)/2$. $d+2$ of these symmetries correspond to the translations along each direction (since the metric does not depend explicitly on any of the coordinates x^μ, each ∂_μ is a Killing vector). The rest of the symmetries are $(d+2)(d+3)/2 - (d+2) = (d+2)(d+1)/2$ transformations that preserve the form of the metric: $M^{-1} g M = g \implies M \in SO(2,d)$ by definition. Not all of these symmetries survive as we move onto the submanifold, we defined above. Clearly the $d+2$ traslations $x^\mu \to x^\mu + a^\mu$ do not preserve the hypersurface 4.3. But the AdS space inherits the $SO(2,d)$ symmetry from $\mathbb{R}^{2,d}$. To show this, one can consider two vectors with components v_1^a, v_2^a tangent to a point in the AdS space. These vectors also lie in the tangent space to the same point in the embedding space. Now if we transform them with $SO(2,d)$ in the embedding space, we have:

$$\eta_{\mu\nu}v_1^\mu v_2^\nu = \eta_{\mu\nu}v_1'^\mu v_2'^\nu, \tag{4.4}$$

where the primed components correspond to the transformed vectors. Now we know that the induced metric in the AdS space, $h_{\mu\nu}$ for vectors which are both tangent to the embedding space and the AdS space has the property:

$$\eta_{\mu\nu}v_1^\mu v_2^\nu = h_{\mu\nu}v_1^\mu v_2^\nu. \tag{4.5}$$

Furthermore we know that after the transformation these vectors are still tangent to the AdS space, because transformation is nothing but a relabeling. Thus by 4.4 and 4.5 AdS space is also $SO(2,d)$ invariant. This invariance does not depend on the embedding and one can find all Killing vectors directly from 4.1 and show that they satisfy the $so(2,d)$ algebra. The maximum number of isometries in a $d+1$-dimensional space is $(d+2)(d+1)/2$, which is exactly the number of generators for the $so(2,d)$ algebra, thus we have found all the isometries of the submanifold. The significance of this invariance is that this algebra is exactly the Lie algebra corresponding to the conformal group in $(d-1)+1$ dimensions. Now to show that the submanifold 4.3 is indeed the AdS spacetime we find the induced metric. We showed that this submanifold has the $SO(2,d)$ symmetry, so it has the $SO(d)$ symmetry as a subgroup. So one can utilize the unit S^{d-1} coordinates $\omega^i(i=1,...,d)$ to write the AdS_{d+1} metric:

$$x^0 = R_{AdS}\cosh\rho\,\cos\tau,\quad x^{d+1} = R_{AdS}\cosh\rho\,\sin\tau,\quad x^i = R_{AdS}\sinh\rho\,\omega^i, \tag{4.6}$$

where $\rho \geq 0$ and $2\pi > \tau \geq 0$. This is the global coordinates for AdS_{d+1} and they cover the whole submanifold 4.3. The metric becomes

$$ds^2 = R_{AdS}^2\big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2\big), \tag{4.7}$$

which is the same metric as 4.1, but with a small difference which comes from the domain of definition of the coordinate τ in 4.6. Let us look at the surfaces of constant τ for the submanifold 4.3. For these surfaces the induced metric is:

$$ds^2 = R_{AdS}^2\big(d\rho^2 + \sinh^2\rho d\Omega_{d-1}^2\big). \tag{4.8}$$

This is the metric for the hyperbolic space H^d. For the case of AdS_3 we shall carefully analyze these surfaces but for now we have to realize that one can show that there exists a homeomorphism between H^d and $\mathbb{R}^d$, thus the topology of surfaces of constant τ for our submanifold is that of $\mathbb{R}^d$. This means that globally the topology of the submanifold is that of $S^1 \times \mathbb{R}^d$, where S^1 is the compact dimension where the periodic τ lives in. In a spacetime where the timelike direction is S^1 one may argue that there might be possible to form closed timelike curves and cause problems for causality. Thus it is desired to remove these curves if possible. This may be achieved due to the fact that the metric components in 4.7 are time independent and the periodicity in time is invisible. One simply unwraps all of the time circles S^1 and extends them in the line of real numbers. This means that now we have $\tau \in \mathbb{R}$. Geometrically this accounts to making an infinite number of turns around the submanifold 4.3 in its time direction. To avoid the periodic identification of points in this direction one discards the submanifold model and introduces a new spacetime. This means that one discards the whole constructing of 4.7 based on employing the embedding space $\mathbb{R}^{2,d}$ and constructing the submanifold on it. Instead one defines the manifold $\mathbb{R}^{d+1}$ with the metric 4.7 where now $\tau \in \mathbb{R}$. This spacetime is called the "universal covering space of the AdS" space denoted by $CAdS$, which is exactly the same thing we defined at the beginning of this section. Now the reason for this definition and it's topology and name is more clear. Finally one must note that usually people regard $CAdS$ and AdS spacetimes the same thing and call them AdS, meaning they don't consider the construction of submanifold 4.3 from the beginning. From now on whenever we say AdS we mean the spacetime $CAdS$. Now it is time to introduce other coordinates in the AdS spacetime.

Another useful coordinate system for this space is the Poincare coordinates which are defined by:

$$\begin{aligned} x^0 &= \frac{R_{AdS}r}{2}\big((x^i)^2 - t^2 + \frac{1}{r^2} + 1\big), \\ x^{d+1} &= R_{AdS}rt, \\ x^i &= R_{AdS}rx^i \quad (i = 1, ..., d-1), \\ x^d &= \frac{R_{AdS}r}{2}\big((x^i)^2 - t^2 + \frac{1}{r^2} - 1\big). \end{aligned} \tag{4.9}$$

Then the metric becomes

$$\boxed{ds^2 = R_{AdS}^2\left(r^2(-dt^2 + \sum_i^d d(x^i)^2) + \frac{dr^2}{r^2}\right).} \tag{4.10}$$

Another version of the Poincare coordinates can be obtained by $z := 1/r$,

$$\boxed{ds^2 = R_{AdS}^2\frac{dz^2 - dt^2 + \sum_i^d d(x^i)^2}{z^2}.} \tag{4.11}$$

One must note that Poincare coordinates do not cover the whole AdS spacetime.

4.2 $(2+1)-$dimensional Gravity

It is fruitful to discuss the significance of the (2 + 1)-dimensional gravity before moving further. General relativity in the (3 + 1)-dimensions which is the assumed dimension for our observable world is an accurate and strong theory but it is very complex, both at the classical and quantum level.

The usual approach in such situations is to look for a simplified version of the theory which provides insights and is not trivial at the same time. Luckily for the case of gravity, reduction of one spatial dimension seems to give us the toy model we desire.

As we shall see, (2+1)-dimensional theory has certain properties that are of importance. Firstly the vacuum solutions in (2 + 1)-dimensions are all constant curvature and thus maximally symmetric (in (3+1)-dimensions this is not the case, for example the Schwarzschild solution is a prime example of non-constant curvature vacuum solution). Secondly by topological consideration the spacetime still contains black holes which saves the theory from its apparent triviality. The details of these properties will be discussed as we move further.

4.2.1 Curvature in $(2+1)-$dimensional Gravity

From geometry one knows that for a semi-Riemannian manifold (M, g) the number of the independent parameters in the Riemann's curvature tensor is $\frac{n^2}{12}(n^2-1)$. One also knows that this tensor can be decomposed into a trace-part, called the Ricci tensor $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ and a traceless conformally invariant part, called the Weyl tensor $C_{\mu\nu\alpha\beta}$:

$$R_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + \frac{2}{n-2}(g_{\alpha[\mu}R_{\nu]\beta} + R_{\alpha[\mu}g_{\nu]\beta}) - \frac{2R}{(n-1)(n-2)}g_{\alpha[\mu}g_{\nu]\beta}. \tag{4.12}$$

In general relativity, the Weyl curvature is the only part of the curvature that exists in free space—a solution of the vacuum Einstein equation—and it governs the propagation of gravitational waves through regions of space devoid of matter. So one can look at it as a tool to realize the gravitation information in a source-free scenario.
The first thing to realize in $(2+1)$-dimensions is that the number of parameters for Riemann tensor are 6. It is exactly the number of independent parameters of the symmetric Ricci tensor. In fact one can readily show that $C_{\mu\nu\alpha\beta}$ vanishes identically in this case. This is interpreted as no pure gravitational degree of freedom in $(2+1)$-dimensional Einstein gravity.
Next we can talk about the equations of motion. In the case of $(2+1)$-dimensions, the Einstein-Hilbert action provides another interesting feature. For the vacuum, this action is given by:

$$S[g] = \frac{1}{16\pi G}\int_M d^3x\sqrt{-g}(R+2\Lambda), \tag{4.13}$$

where R is the Ricci scalar and Λ is the cosmological constant. In this case the equations of motion are $R_{\mu\nu} + (\Lambda - \frac{R}{2})g_{\mu\nu} = 0$. by taking the trace one has $R = 6\Lambda$. Thus on shell (when the equations of motion are obeyed for the metric), the Ricci tensor is completely determined by the metric tensor:

$$R_{\mu\nu} = (\frac{R}{2} - \Lambda)g_{\mu\nu} \implies R_{\mu\nu} = 2\Lambda g_{\mu\nu}. \tag{4.14}$$

Now if we plug the expression above into 4.12 and remember that the Weyl tensor vanishes for $(2+1)$-dimensions, we have:

$$R_{\alpha\beta\mu\nu} = \Lambda(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\nu}). \tag{4.15}$$

This is a significant result, because it shows that every vacuum solution of the $(2+1)$-dimensional gravity has constant curvature or is a locally homogeneous space. These spaces are all maximally symmetric meaning that they have the maximum number of Killing vectors as mentioned earlier. The eq. 4.15 can be solved for $g_{\mu\nu}$ for different signs of Λ. By solving in each local chart we find that the ion each chart the metric can be AdS_3, dS_3 or flat. Thus depending on the sign of the cosmological constant, there can be 3 possibilities for these spaces:

(1) If $\Lambda > 0$ the solutions are locally isometric to the $(2+1)$-dimensional de Sitter space.
(2) If $\Lambda = 0$ the solutions are locally isometric to the $(2+1)$-dimensional Minkowski (flat) space.
(3) If $\Lambda < 0$ the solutions are locally isometric to the $(2+1)$-dimensional anti-de Sitter space.

There can be an ambiguity in the meaning of locally isometric, so it can be good to restate its definition.

Defintion (Isometry and Local Isometry): Let $(M.g)$ and (M', g') be two semi-Riemannian manifolds, and let $f : M \to M'$ be a diffeomorphism. Then f is called an isometry (or isometric isomorphism) if $g = f^*g'$, where f^*g' denotes the pullback of the rank $(0, 2)$ metric tensor g' by f. If f is a local diffeomorphism (From an open set $U \in M$ to an open set $V \in M'$) such that $g = f^*g'$, then f is called a local isometry.

From the definition of the local isometry we realize that the classification above implies that at each open set on our spacetime we can find a map that maps our spacetime to an open set in the AdS spacetime such that this map is an isometry. Thus the vacuum solutions for the $(2+1)$-dimensional gravity are simply classified, but this does not trivialize the theory. As we shall see, we can still have black holes due to the topological properties. For our purposes we will further discuss only one of the cases above which is the $(2 + 1)$-dimensional anti-de Sitter space denoted by AdS_3.

4.2.2 AdS_3 Spacetime and Geodesics

So far we discussed the AdS_{d+1} and $(2 + 1)$-dimensional gravity with some detail. It is useful to further discuss on of the simplest non-trivial cases of AdS_{d+1} which is AdS_3 that is as we know from the last section the only negative curvature solution for the vacuum $(2 + 1)$-dimensional gravity. One can introduce a length scale $R_{AdS} \in \mathbb{R}^+$ such that $\Lambda = -1/R^2_{AdS}$. We call this scale R_{AdS} because it will turn out to be the AdS radius that we introduced earlier. Now one can check that the solution for the vacuum Einstein equations is:

$$ds^2 = -\Big(1 + \frac{r^2}{R^2_{AdS}}\Big)d\tilde{t}^2 + \Big(1 + \frac{r^2}{R^2_{AdS}}\Big)^{-1} dr^2 + r^2 d\theta^2, \tag{4.16}$$

where $\tilde{t} \in \mathbb{R}$ is a timelike coordinate, $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi]$. This is the AdS_3 space in the static coordinate. The scale R_{AdS} determines the distance at which the curvature sets in. At $(r \ll R_{AdS})$, the metric above is simply flat, and as $(r \gg R_{AdS})$, we get $ds^2 \sim \frac{R_{AdS}}{r^2} dr^2 + \frac{r^2}{R_{AdS}}(-d\tilde{t}^2 + R^2_{AdS} d\theta^2)$. So the behavior of the boundary is similar to that of a cylinder at the large r. We will further discuss this later on when we discuss the Poincare disks. Now if we do the change $r = R_{AdS}\, sinh\, \rho$ and $\tau = \tilde{t}/R_{AdS}$ we have:

$$ds^2 = R^2_{AdS}\big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\theta^2\big). \tag{4.17}$$

This is the usual global coordinates representation of the metric that we had earlier. The non-zero Christoffel symbols for this coordinates are:

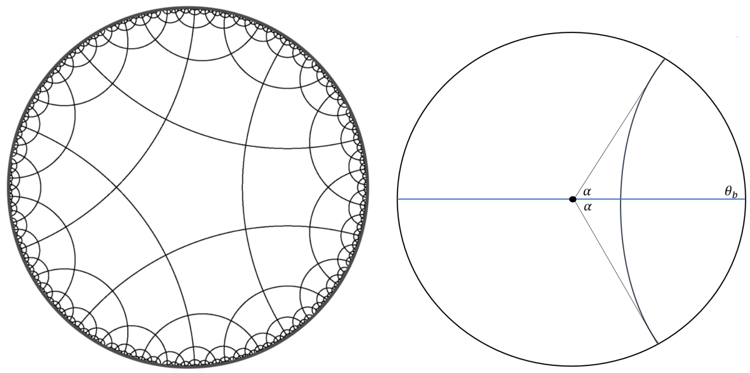
$$\Gamma^\tau_{\tau\rho} = \Gamma^\tau_{\rho\tau} = \tanh\rho, \quad \Gamma^\rho_{\tau\tau} = -\Gamma^\rho_{\theta\theta} = \sinh\rho\, \cosh\rho, \quad \Gamma^\theta_{\rho\theta} = \Gamma^\theta_{\theta\rho} = \coth\rho. \tag{4.18}$$

In the Poincare coordinates for AdS_3 we have:

$$ds^2 = \frac{R^2_{AdS}}{z^2}(-dt^2 + dx^2 + dz^2). \tag{4.19}$$

And in this case one has:

$$\Gamma^t_{tz} = \Gamma^t_{zt} = \Gamma^x_{xz} = \Gamma^x_{zx} = \Gamma^z_{tt} = \Gamma^z_{zz} = -\Gamma^z_{xx} = -\frac{1}{z}. \tag{4.20}$$



(a) (b)

Figure 6: (a) A depiction of the compactified Poincare disk with its geodesics. (b) A single geodesic in the compactified Poincare disk and relevant quantities

If we restrict ourselves to constant t in the global coordinates or to constant τ in the Poincare coordinates, we will have a two-dimensional subspace of AdS_3 called the Poincare Disk. This space is of interest in the hyperbolic geometry as we mentioned earlier in eq. 4.8, but for our purposes it can be viewed as a subspace of AdS_3 which is from the perspective of an observer who sees all the events on this disk simultaneously. From 4.11 one can see that metric in global coordinates for this space is given by:

$$ds^2 = R_{AdS}^2(d\rho^2 + \sinh^2\rho\, d\theta^2), \tag{4.21}$$

where $0 \leq \rho < \infty$ and $0 \leq \theta < 2\pi$. Consider a radial segment ($d\theta = 0$) emanating outwards from the origin. The length of that segment is given by:

$$s = \int ds = \int R_{AdS} d\rho = R_{AdS}\rho + const, \tag{4.22}$$

where the cosntant is manually set equal to 0. An interesting point that one realizes by looking at this integral is that there is a fundamental length scale in the AdS space, namely R_{AdS}. For this reason Gauss once wistfully remarked: ”I have sometimes in jest expressed the wish that Euclidean geometry is not true. For then we would have an absolute a priori unit of measurement”. Clearly, this integral will diverge as ρ is permitted to increase without bound. Hence the Poincare disk represents an infinte two-dimensional space. Nevertheless, it is depicted (as it’s name would suggest) by a finite disk, though one is supposed to imagine that the disk’s boundary is infinitely far away from any point in its interior. These points at the boundary are called ideal points, omega points, vanishing points, or points at infinity. The next point of interest in the Poincare disks is the geodesics. Ordinarily in general relativity, the geodesics of principal interest are the timelike and null geodesics, as these are the possible paths which a massive or massless particle could take through spacetime, respectively. However, for purposes of this study, we are primarily interested in a subset of the spacelike geodesics, which are those with constant proper time, that are paths confined to the Poincare disk. For the global coordinates using the Christoffel symbols 4.18 the geodesic equation in the Poincare disk is:

$$\begin{aligned}&\frac{d^2\rho}{d\lambda^2} - \sinh\rho\,\cosh\rho\left(\frac{d\theta}{d\lambda}\right)^2 = 0,\\&\frac{d^2\theta}{d\lambda^2} + 2\coth\rho\left(\frac{d\rho}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right) = 0,\end{aligned} \tag{4.23}$$

where λ is an affine parameter, meaning it is some linear function of the distance, i.e. $\lambda = as + b$. The curve which satisfies these equations is:

$$\tanh\rho\,\cos(\theta - \theta_b) = \cos\alpha, \tag{4.24}$$

where the meaning of θ_b and α will become clear shortly. If one defines the following change of coordinates on the Poincare disk:

$$\rho = \ln\left(\frac{1+\tilde{r}}{1-\tilde{r}}\right), \tag{4.25}$$

where $0 \le \tilde{r} < 1$, then the infinite disk would be compactified to a disk of radius 1, on which the points at infinity lie at the boundary of the disk. If we do this change of coordinates in the 4.24 we would have:

$$\begin{aligned}&\frac{2\tilde{r}}{\tilde{r}^2+1}\big(\cos\theta\,\cos\theta_b + \sin\theta\,\sin\theta_b\big) = \cos\alpha\\&\implies 2\tilde{x}\cos\theta_b + 2\tilde{y}\sin\theta_b = (\tilde{x}^2 + \tilde{y}^2 + 1)\cos\alpha\\&\implies \tilde{x}^2 - \frac{2\tilde{x}\cos\theta_b}{\cos\alpha} + \tilde{y}^2 - \frac{2\tilde{y}\sin\theta_b}{\cos\alpha} = -1 = \tan^2\alpha - \frac{1}{\cos^2\alpha} = \tan^2\alpha - \frac{\sin^2\theta_b}{\cos^2\alpha} - \frac{\cos^2\theta_b}{\cos^2\alpha}\\&\implies \left(\tilde{x} - \frac{\cos\theta_b}{\cos\alpha}\right)^2 + \left(\tilde{y} - \frac{\sin\theta_b}{\cos\alpha}\right)^2 = \tan^2\alpha,\end{aligned} \tag{4.26}$$

where we defined $\tilde{x} = \tilde{r}\cos\theta, \tilde{y} = \tilde{r}\sin\theta$. The equation above shows that geodesics in the compactified Poincare disk are given by circular arcs which intersect the disk boundary at a pair of omega points such that a line tangent to these circular arcs is perpendicular to a line tangent to the Poincare disk's boundary. θ_b is the angular coordinate of the boundary point that lies on the unique radial line bisecting the circular arc, and α is the angular separation between θ_b and either omega point which intersects the arc (Figure 6). Next in the Poincare coordinates we can again look at the geodesic equations using the Christoffel symbols 4.20:

$$\begin{aligned}\frac{d^2x}{d\lambda^2} - \frac{2}{z}\left(\frac{dx}{d\lambda}\right)\left(\frac{dz}{d\lambda}\right) &= 0,\\\frac{d^2z}{d\lambda^2} + \frac{1}{z}\left[\left(\frac{dx}{d\lambda}\right)^2 - \left(\frac{dz}{d\lambda}\right)^2\right] &= 0.\end{aligned} \tag{4.27}$$

For solving these equations write x and z each as a function of g which itself is a function of the distance s (remember that $\lambda = as + b$):

$$x(g) = \frac{l}{2}\cos g, \quad z(g) = \frac{l}{2}\sin g, \quad g(s) = 2\tan^{-1}(e^s), \tag{4.28}$$

where l is related to the quantity α by $\alpha = \pi l/L$, and L is the circumference of the compacitified Poincare disk. To find the distance between two interior points A and B in the Poincare disk, one has to draw the unique geodesic that intersects both of them and extend the geodesic to the boudnary. Let the omega points where this geodesic intersects the boundary be denoted by P and Q (Figure 7). Then from the hyperbolic geometry it is known that a good measure of distance is

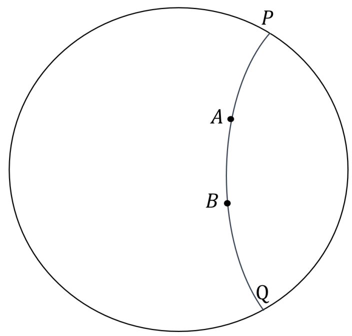


Figure 7: A sample geodesic in a Poincare disk with omega points P and Q and interior points A and B

defined in terms of arc lengths AP, BQ, AQ, and BP as follows:

$$distance(A, B) := \left| \ln \left(\frac{AP.BQ}{AQ.BP} \right) \right|. \tag{4.29}$$

Using the definition of the distance above we claim that, The distance between any point in the Poincare disk and any ideal point is infinite.
To prove our claim we try to calculate the distance between interior point A and ideal point Q by making the substitution $B = Q$ in 4.29:

$$distance(A, Q) := \left| \ln \left(\frac{AP.QQ}{AQ.QP} \right) \right|. \tag{4.30}$$

QQ is not actually a line segment, but we can think of it as the limiting case in which $B \to Q$. Hence the numerator of this fraction is infinitesimal (approaching zero in the limit), while the denominator is positive. Hence the absolute value of the logarithm is increasing without bound in this limit. If both points are ideal ($A = P$), then the numerator is still zero and the denominator is still non-zero, and hence we reach the same result.
The argument above shows that the Poincare disk has infinite spatial extent. This is important in that it means that the length of all geodesics is infinite. As we will see later, in order to speak of the length of a geodesic, we must impose a manual cutoff.

4.2.3 AdS_3 Isometries, BTZ Black Hole

Next topic to cover is the isometries of AdS_3. From our discussion of AdS_{d+1}, we know that the group of isometries that the Killing vectors form for the case of AdS_3, is the $SO(2, 2)$. Let us consider the AdS_3 again as a submanifold, $-v^2 - u^2 + x^2 + y^2 = -R_{AdS}$ in $\mathbb{R}^{2,2}$ as we did in 4.3 and denote $x^a = (v, u, x, y)$. The generators of $so(2, 2)$ isometry in the $\mathbb{R}^{2,2}$ are given by the Killing vectors:

$$J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}, \tag{4.31}$$

where we defined the $x_a = \eta_{ab}x^b$. This definition of x_a is just a mere combination of the coordinates and does not necessarily mean that x^a is a vector. In detail this means that we have:

$$\begin{aligned} J_{01} &= v\partial_u - u\partial_v, \quad J_{02} = x\partial_v + v\partial_x, \\ J_{03} &= y\partial_v + v\partial_y, \quad J_{12} = x\partial_u + u\partial_x, \\ J_{13} &= y\partial_u + u\partial_y, \quad J_{23} = y\partial_x - x\partial_y. \end{aligned} \tag{4.32}$$

To find the representation in the submanifold representing the AdS_3 one expands the x^a in 4.6 and finds that $J_{01} = \partial_\tau$ generates the time evolution, whereas $J_{12} = \partial_\phi$ generates rotations. Form of other Killing vectors in the submanifold coordinates will not be important for us. One must also note that one must take $\tau \in \mathbb{R}$ to avoid closed causal curves due to our discussions in the beginning of the chapter.

As we mentioned earlier, appearance of the black holes in $(2+1)$-dimensional gravity is an important feature that makes it interesting to study. The black hole solutions for (2+1)-dimensional gravity were first found by Banados, Teitelboim and Zanelli (BTZ) [18]. These solutions can be found by certain topological identifications in AdS_3. We will first present the BTZ black holes and discuss their properties and then briefly discuss how they can be found from identifications of points in AdS_3.
To begin, let us set aside the global homogeneous spaces (global is important in this context, because local behavior would not give anything new since all of the solutions are locally homogeneous as we seen in section 3.2.1) and look for more general vacuum solutions. Also let us further demand that these geometries asymptote to AdS_3 when approaching to spatial infinity. These spacetimes are called asymptotically AdS_3. These kind of spaces can contain a black hole region, meaning a region which no causal curve can escape from. One has to remember that this does not contradict our analysis that all solutions of $(2+1)$-dimensional vacuum gravity are locally AdS_3 for negative cosmological constant. This means that even though locally around each point one can find an isometry between the neighborhood of that point in these spacetimes and an open set in AdS_3 but topologically the spaces can differ. A prime example of this situation is the flat space and a cylinder which are isometric but not topologically equivalent. Further as we shall see the spacetime will have no curvature singularity. It is a black hole because it admits an event horizon which will turn out to protect a "causal singularity".

Now we first present the BTZ solution. Let us consider a set of Boyer-Lindquist-like coordinates $(\tilde{t}, r, \theta)$, where $\tilde{t}$ is an asymptotically timelike coordinate, r is an asymptotically radial coordinate, and θ is a polar angle indified as $\theta \sim \theta + 2\pi$. Imposing that spacetime is asymptotically AdS_3 when $r \to \infty$, and that the spacetime is stationary and axisymmetric (meaning it possesses the killing vectors $\partial_{\tilde{t}}$ and ∂_θ), a natural ansatz to consider is

$$\boxed{ds^2 = -N^2(r)d\tilde{t}^2 + \frac{dr^2}{N^2(r)} + r^2(d\theta + N^\theta(r)d\tilde{t})^2.} \tag{4.33}$$

This solution with appropriate $N(r)$ and $N^\theta(r)$ was found for the first time in 1992 by Banados, Teitelboim, and Zanelli, and describes in some range of the parameters, as we are about to show, the so-called BTZ black hole. The functions in the metric above are fixed such that:

$$\begin{aligned} N^2(r) &= -8MG + \frac{r^2}{R_{AdS}^2} + \frac{16G^2J^2}{r^2} \quad (Lapse\ \ function), \\ N^\theta(r) &= -\frac{4GJ}{r^2} \quad (Angular\ \ dragging). \end{aligned} \tag{4.34}$$

One can explicitly verify that the BTZ solution has 2 Killing vectors $\partial_{\tilde{t}}$ and ∂_θ since the metric coefficients depend only on r. But this solution is not static, because the term $d\tilde{t}d\theta$ is not invariant under time reversal. Moreover the quantities M and J are charges associated to $\partial_{\tilde{t}}$ and ∂_θ. Furthermore one can show that the only symmetries for this spacetime are the two that we mentioned, meaning that unlike AdS_3 this spacetime is not maximally symmetric.

To discuss the properties of this spacetime we first note that the global AdS_3, 4.16 is recovered if $N^\theta = 0 \implies J = 0$ and $M = -1/8G$. By taking the limit $r \to \infty$ one can show that indeed this solution is asymptotically AdS_3. Further if we calculate the Ricci scalar we see that $R = -6/R_{AdS}^2$ everywhere. This means that the solution does not have any curvature singularity, in accordance with our expectation that it must be locally AdS_3 everywhere.
The set of roots for the lapse function, $N(r)^2$ are important in our analysis. By only considering the positive values of r we see that:

$$N(r)^2 = 0 \implies r = r_\pm = R_{AdS}\sqrt{4GM}\sqrt{1 \pm \sqrt{1 - \left(\frac{J}{MR_{AdS}}\right)^2}}. \tag{4.35}$$

Let us denote by $\mathcal{H}_\pm$ the surfaces $\{x^\mu | r = r_\pm\}$. These surfaces exists when the square roots are well defined meaning:

$$|J| \leq MR_{AdS}, \ \ M > 0. \tag{4.36}$$

This limits the spectrum of mass and spin of the allowed black holes. With an analysis similar to that of the Kerr black holes one can show that the region $r < r_+$ is a black hole region, meaning that no causal signal inside them can escape to $r > r_+$, so indeed these solutions contain black holes.
These black holes are said to be extremal if $|J| = MR_{AdS}$ or equivalently $r_+ = r_-$. Another important thing to note is that at $r = 0$ the Killing vector ∂_θ is no longer necessarily spacelike. Given that the coordinate θ is periodic this can lead to closed causal curves, and thus we regard $r = 0$ as a causal singularity. Luckily because $r = 0 < r_+$ we have a kind of cosmic censorship meaning that the event horizon protects protects the outside observers from this causal singularity.
One can show that the surface $\mathcal{H}_+$ is in fact a Killing horizon corresponding to the Killing vector $\xi = \partial_{\tilde{t}} + \Omega_H \partial_\theta$, where $\Omega_H = -N^\theta(r_+) = \frac{r_-}{r_+} R_{AdS}$. So we have an event horizon which is also a Killing horizon, meaning that the Hawking's rigidity theorem holds.
Similar to the Kerr spacetime one can show that for this Killing horizon on can calculate the surface gravity which will be $\kappa = \frac{(r_+^2 - r_-^2)}{R_{AdS} r_+}$. Given this surface gravity the Hawking's temperature is readily given by:

$$T_H = \frac{\kappa}{2\pi} = \frac{r_+^2 - r_-^2}{2\pi R_{AdS} r_+}. \tag{4.37}$$

Finally we want to briefly discuss the derivation of the BTZ solutions. Any Killing vector ξ^μ generates a 1-parameter subgroup of isometries of AdS_3. This subgroup's action on the points of

the spacetime can be expressed by the exponential map: $P \to e^{s\xi}P$, where s is continuous parameter and P is a point in AdS_3 spacetime manifold. If we restrict s to discrete values $k\Delta s$ where $k \in \mathbb{Z}$ and Δs is a basic step conventionally fixed as 2π, we are left with a so called identification subgroup. The identified quotient space is obtained by identifying points that belong to a given orbit of the identification subgroup. This means that we define a relation between points in the manifold such that $P \sim P'$ iff $P' = e^{s^{2k\pi}}P$. Then we can easily show that this relation is an equivalence relation. The equivalence classes of this relations, $[P] = \{P'|P' = e^{s^{2k\pi}}P\}$ then are defined to be points in the quotient space.
Since ξ^μ is a Killing vector, this quotient process leads to a continuous spacetime and the quotient space inherits from AdS_3 space a well defined metric which has constant negative curvature. It is because if we choose a coordinate system $(\tilde{t}, r, \theta)$ in which $\xi = \partial_\theta$ and $\theta \in \mathbb{R}$, then because of the Killing equation:

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 0 = \partial_\theta g_{\mu\nu} = 0 \implies g_{\mu\nu}(\tilde{t}, r, \theta) = g_{\mu\nu}(\tilde{t}, r). \tag{4.38}$$

Thus in this well chosen coordinate, metric does not depend on the coordinate θ, so the quotient space indeed inherits the metric from the original space, and the quotient space remains a solution for the Einstein's equations.
The identification process makes the curves joining two points that are on the same orbit to be closed in the quotient space. In order to preserve causality in the quotient space and prevent the creation of closed causal curves, a necessary condition (but not sufficient in general) is that the Killing vector must be spacelike: $\xi^\mu \xi_\mu > 0$. Now let us see what happens particularly for the BTZ black hole.
BTZ showed in their paper [18] that the non-extremal BTZ black hole solutions are obtained by making identifications in AdS_3 by the discrete group generated by the Killing vector:

$$\xi = \frac{r_+}{R_{AdS}} J_{12} - \frac{r_-}{R_{AdS}} J_{03}, \tag{4.39}$$

where as we have seen before the J_{ab} are the Killing vectors of AdS_3 belonging to the $so(2,2)$ algebra. Then by finding a suitable coordinate system $(\tilde{t}, r, \theta)$ such that $\xi = \partial_\theta$, BTZ showed that the AdS_3 metric would take the form 4.33 but with $\theta \in \mathbb{R}$. Thus simply by making the identification for θ coordinate we will arrive at the BTZ solution for which $\theta \in [0, 2\pi]$. This explicitly shows that the BTZ solution is locally isometric to AdS_3.

Before closing this section, let us look at the BTZ solution when $J = 0$, then we can readily check that we have:

$$ds^2 = -(\frac{r^2 - r_+^2}{R_{AdS}^2})d\tilde{t}^2 + \frac{R_{AdS}^2}{r^2 - r_+^2}dr^2 + r^2 d\theta^2. \tag{4.40}$$

One thing of note here is that in this case the BTZ black hole is not only stationary but also static. If we move to Euclidean signature we have:

$$ds^2 = (r^2 - r_+^2)d\tau_{euc}^2 + \frac{R_{AdS}^2}{r^2 - r_+^2}dr^2 + r^2 d\theta^2, \tag{4.41}$$

where we denoted the imaginary time by τ_{euc} to differ it from the τ we used before in the global coordinates for the AdS spacetime. Moreover to avoid encountering a conical singularity we must

identify the imaginary time so that: $\tau_{euc} \sim \tau_{euc} + \frac{2\pi R_{AdS}}{r_+}$, where we used the inverse of the temperature we derived in 4.37.
If we do a modular transformation in the boundary CFT for the BTZ black hole so that:

$$r = r_+ \cosh\, \rho', \quad \tau_{euc} = \frac{R_{AdS}}{r_+}\theta', \quad \theta = \frac{R_{AdS}}{r_+}\tau'. \tag{4.42}$$

Then the metric 4.41 becomes the 4.17 with the euclidean signature, such that $\tau' \sim \tau' + \frac{2\pi r_+}{R_{AdS}}$. This means that the Euclidean BTZ black hole at temperature T is equivalent to thermal AdS_3 at temperature $1/T$.

4.3 RT Proposal

According to Ryu and Takayanagi's proposal the entropy S_A in $CFT_{(d-1)+1}$ can be computed from the following area law

$$\boxed{S_A = \frac{Area(\gamma_A)}{4G_N^{d+1}},} \tag{4.43}$$

where the manifold γ_A is the $d-1$ dimensional static (at a constant time) minimal surface (surface for which the area is an extremum value) in the AdS_{d+1} dual to the original $CFT_{(d-1)+1}$, whose boundary is given by ∂A. If there are several of these minimal surfaces one chooses the one with the least area. Furthermore γ_A needs to be homologous to A meaning that they have to fully enclose some region in the bulk slice, i.e. $\gamma_A \cup A = \partial\Sigma$ for some bulk slice Σ. G_N^{d+1} is the $d+1$ dimensional Newton constant. One has to note that this prescription applies only to static spacetimes, where one can choose any time slice and implement the above formula.

In this chapter we will not attempt to prove this proposal, but rather try to justify it by calculating certain examples and comparing them with the results of the previous chapter. In the next chapters we will prove this result and some of it's generalizations.

4.3.1 Entanglement Entropy in $2D$ CFT from AdS_3

Consider a CFT on $\mathbb{R} \times S^{d-1}$ (A finite length system in $d-1$ spatial dimensions) and suppose a subsystem A has a $(d-2)-$dimensional boundary $\partial A \in S^{d-1}$. Now, let γ_A be a unique $(d-1)-$dimensional minimal surface in AdS_{d+1} with boundary ∂A. In the $d = 2$ case, this model can be well represented by a an upright cylinder whose boundary represents the CFT_2 space and whose interior or bulk represents AdS_3. The cylinder's height is the (proper) time dimension, with horizontal circular cross-sections representing Poincare disks. From the differential geometry, it is known that a minimal curve on a two-dimensional surface is in fact a geodesic curve. Let L denote the circumference of the cylinder, which is also the length of the CFT at a fixed time which we studied in the chapter 2. The two-dimensional boundary is divided into two regions A and B along surfaces of constant θ. let l denote the length of subsystem A ($l \leq L$). Then the angular extent of A is $2\pi l/L$ while for B it is $2\pi(1 - l/L)$. These quantities can easily be translated into the AdS language of the section 4.2.2: subsystem A is centered around $\theta = \theta_b$ and has an angular entent of

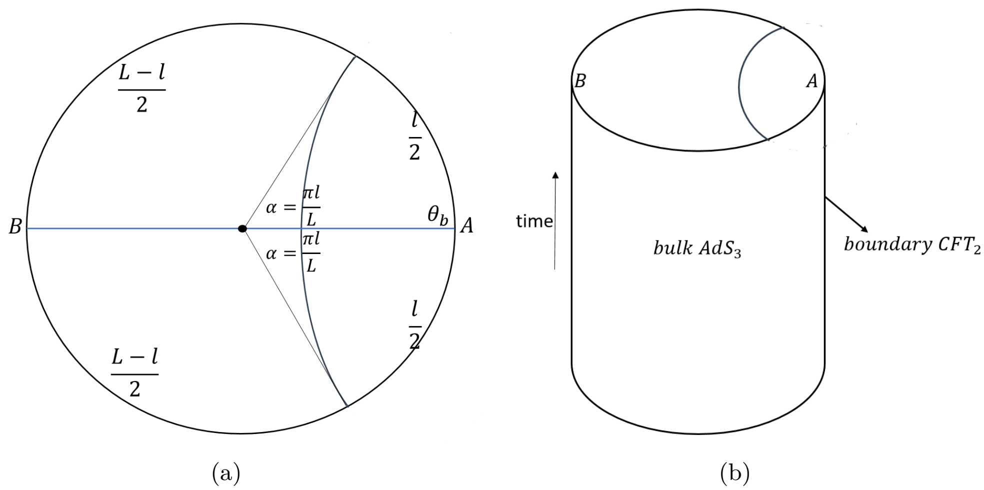


Figure 8: (a) A depiction of the compactified Poincare disk the relevant boundary regions and quantities. (b) cylindrical depiction of the AdS_3/CFT_2 model.

2α. Therefor, $\alpha = \pi l/L$ (Figure 8).
As we argued in the section 4.2.2, at the boundary $\rho \to \infty$ or $\tilde{r} \to 1$ of the AdS_3, the length of the geodesics diverge. Furthermore the metric itself is divergent as well. This would as we shall see imply that the entropy of the CFT interval is also infinite. To avoid this we manually impose a cutoff on ρ: $\rho \leq \rho_0$, where ρ_0 is a large but finite constant. This approximates the boundary as a region of large but finite size. This procedure corresponds to the ultra violet (UV) cutoff in the CFT. This cutoff in CFT is associated with the lattice spacing a, with:

$$e^{\rho_0} \sim L/a, \tag{4.44}$$

where as we mentioned before L is the total length of the system and a is the lattice spacing.
Moreover one has the Brown-Henneaux's [19] relation in AdS/CFT which relates the central charge in the CFT to the AdS radius:

$$c = \frac{3R_{AdS}}{2G_N^{(3)}}. \tag{4.45}$$

The holographic principle tells us that true physical degrees of freedom of the gravitational theory in some region is represented by its boundary of that region. This is well-known in the black hole geometries and it leads to the celebrated area law of the Bekenstein-Hawking entropy. In the context of AdS/CFT correspondence degrees of freedom in AdS_{d+1} space are represented by its boundary of the form $\mathbb{R} \times S^{d-1}$, where the dual conformal field theory lives. We can calculate the degrees of freedom for the case we explained at the beginning of the section. For this we apply the area law in three-dimensional spacetimes to the boundary in whole boundary in the AdS_3:

$$N_{dof} \sim \frac{Length\ of\ the\ boundary\ CFT_2}{4G_N^{(3)}}. \tag{4.46}$$

Next we remember that we imposed a cutoff thus the boundary lives at $\rho = \rho_0$, and to calculate the length we have to use the Poincare disk metric with $d\rho = 0$:

$$L_{CFT_2} = \int_0^{2\pi} R_{AdS} \sinh \rho_0 \, d\theta = 2\pi R_{AdS} \sinh \rho_0. \tag{4.47}$$

finally we use the fact that ρ_0 is large so we estimate:

$$\sinh \rho_0 \approx \frac{e^{\rho_0}}{2} \approx \frac{L}{2a}, \tag{4.48}$$

where we used 4.44. So using 4.45 we have:

$$N_{dof} \sim \frac{\pi c}{6} \frac{L}{a}. \tag{4.49}$$

The central charge c is roughly proportional to the number of fields. The ratio L/a counts the number of independent points in the presence of the lattice spacing a. Therefore the result above agrees with what we expect from the conformal field theory at least up to the unknown numerical coefficient.

Now that we presented the model and imposed a cutoff on our radial coordinate value, we can compute the regulated length s of a geodesic. This length is what we need in order to calculate the entropy using 4.43 in AdS_3, because the static slices of this spacetime are 2-dimensional Poincare disks and as we mentioned before the minimal surface in two dimensions is a geodesic. It is easiest to compute the desired result in Poincare coordinates. Recall from 4.28 that we can write the geodesic as: $(x, y) = l/2(\cos g, \sin g)$, where $g = 2\tan^{-1}(e^s)$. $z = 0$ corresponds to the omega points ($z = 1/r$ and omega points are $r \to \infty$). So we impose the cutoff on $g : \epsilon \leq g \leq \pi - \epsilon$, where ϵ is related to the CFT lattice spacing a by: $\epsilon = 2a/l$. Now since $ds = dg/\sin g$, the regulated length is given by:

$$Length(\gamma_A) = R_{AdS} \int ds = R_{AdS} \int_\epsilon^{\pi-\epsilon} \frac{dg}{\sin\, g} = R_{AdS} \ln \left(\tan \left(\frac{g}{2}\right)\right)_\epsilon^{\pi-\epsilon}. \tag{4.50}$$

Now using the identity, $\tan \frac{x}{2} = \frac{\sin x}{\cos x + 1}$:

$$Length(\gamma_A) = R_{AdS} \ln \left(\frac{\sin(\pi - \epsilon)}{\cos(\pi - \epsilon) + 1} \right) - R_{AdS} \ln \left(\frac{\sin(\epsilon)}{\cos(\epsilon) + 1} \right) = R_{AdS} \ln \left(\frac{\frac{\sin(\pi-\epsilon)}{\cos(\pi-\epsilon)+1}}{\frac{\sin(\epsilon)}{\cos(\epsilon)+1}} \right). \tag{4.51}$$

Then the small ϵ (as we shall see this corresponds to zero temperature, infinite system limit) approximation gives:

$$\begin{aligned} Length(\gamma_A) &= R_{AdS} \ln \left(\frac{\frac{\sin(\pi-\epsilon)}{\cos(\pi-\epsilon)+1}}{\frac{\sin(\epsilon)}{\cos(\epsilon)+1}} \right) = R_{AdS} \ln \left(\frac{\frac{\sin(\epsilon)}{-\cos(\epsilon)+1}}{\frac{\sin(\epsilon)}{\cos(\epsilon)+1}} \right) = R_{AdS} \ln \left(\frac{\cos\epsilon + 1}{-\cos\epsilon + 1} \right) \\ &\approx R_{AdS} \ln \left(\frac{1 - \epsilon^2/2 + 1}{-(1 - \epsilon^2/2) + 1} \right) \approx R_{AdS} \ln \left(\frac{2}{\epsilon^2/2} \right) = 2R_{AdS} ln \left(\frac{2}{\epsilon} \right). \end{aligned} \tag{4.52}$$

Now note that $\epsilon = 2a/l, 2/\epsilon = l/a$. So:

$$Length(\gamma_A) = 2R_{AdS}\ln\left(\frac{l}{a}\right). \tag{4.53}$$

Finally using the RT proposal and 4.45 one has:

$$S_A = \frac{Length(\gamma_A)}{4G_N^{(3)}} = \frac{R_{AdS}}{2G_N^{(3)}}\ln\left(\frac{l}{a}\right) = \frac{c}{3}\ln\left(\frac{l}{a}\right). \tag{4.54}$$

This expression is consistent with 3.28 which is the entropy for an interval of length l in the zero temperature CFT. by taking the $\epsilon \to 0$ above in our approximations we extended the Poincare disks to infinity and thus arrived at the theory with zero temperature and infinite length. One has to note that the limit we took for ϵ was a new assumption that was not present in the cutoff ρ_0 itself. Next we have to try and do the calculations without this further assumption to find the expression for finite length.
A more exact formula for the regulated length s can be found using the global coordinates:

$$\cosh\frac{s}{R_{AdS}} = \big(1 + 2\sinh^2\rho_0 \sin^2(\frac{\pi l}{L})\big). \tag{4.55}$$

In order to calculate the entropy in this case we realize that $s \gg R_{AdS}$, because by imposing the cutoff the length of geodesic with respect to AdS radius is still large. So we have $x \approx ln(2cosh\,x), when\;\; x \to \infty$ and thus:

$$\frac{s}{R_{AdS}} \approx \ln\left(2\cosh\big(\frac{s}{R_{AdS}}\big)\right) = ln\left[2\left(1 + 2\sinh^2\rho_0\sin^2(\frac{\pi l}{L})\right)\right]. \tag{4.56}$$

Now because ρ_0 is large we have:

$$\frac{s}{R_{AdS}} \approx \ln\left[2 + e^{2\rho_0}\sin^2(\frac{\pi l}{L})\right] \implies s \approx R_{AdS}\ln\left(e^{2\rho_0}\sin^2\left(\frac{\pi l}{L}\right)\right), \tag{4.57}$$

where we also dropped the addition of two in the logarithm's argument in the right side expression. Finally we use this regulated length in the RT formula:

$$S_A \approx \frac{R_{AdS}}{4G_N^{(3)}}\ln\left(e^{2\rho_0}\sin^2\left(\frac{\pi l}{L}\right)\right) = \frac{R_{AdS}}{2G_N^{(3)}}\ln\left(e^{\rho_0}\sin\left(\frac{\pi l}{L}\right)\right). \tag{4.58}$$

Now with the use of 4.44 if we postulate the proportionality factor to be $1/\pi$ we have: $e^{\rho_0} = L/(\pi a)$. Considering this relation for cutoff and 4.45 one has:

$$S_A \approx \frac{c}{3}\ln\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right). \tag{4.59}$$

Which is exactly the relation 3.33 for a system with finite length L.

The next case of interest would be the calculation of the entropy of an infinite system at finite temperature. To be more precise we are interested in cases for which $\beta/L \ll 1$, where β is the inverted temperature of the CFT and L is the length of the system. For these cases the gravity

dual to the CFT is described by the Euclidean BTZ black hole. The metric of this space is given by 4.41.
This spacetime is static as we mentioned earlier and also there is an equivalence between Euclidean BTZ and thermal AdS_3. This equivalence is encompassed in 4.42. So to calculate the length of the geodesic in the black hole spacetime we have to transform the coordinates to the AdS_3 in which we know the results.
Now that we have calculated some explicit examples of the RT formula it is a good place to end this chapter and postpone the derivation and further understanding of this formula to the later chapters.

5 Generalized Gravitational Entropy, Prescription of Lewkowycz and Maldacena

After the proposal of Ryu and Takayanagi for the calculation of entanglement entropy, it was natural to ask how one can prove it? There were numerous evidences that the conjecture is right. For example we've shown in the previous chapter it provides the right formula for entanglement entropy in CFT_2 in which we know how to compute it, using the Cardy and Calabrese's method. However no proofs were given except for the case of AdS_3 [20][21]. In these proofs the idea is to use the fact that any constant negative curvature spacetime is locally isometric to AdS_3 so it is natural to expect that these spacetimes are found by findng a quotient of AdS_3 under a certain discrete group of isometries. In fact it can be proven that quotients of AdS_3 under certain groups, known as the Kleinian groups are in correspondence with the constant negative curvature spacetimes [22]. This procedure of finding this quotient map provides a differntial equation for the quotient map, and imposing the trivial monodromy condition on that equation will provide us with a differential equation to find the Renyi entropies. Of course this explanation that we gave is too simplified and only gives the main idea. But we don't explain these proofs further here, since our aim is to explain a more general argument that can explain why RT conjecture is correct.

In this chapter we will review the work of Maldacena and Lewkowycz (LM) [5] in which they proved a generalized gravitational entropy formula for the spaces without $U(1)$ symmetry in the euclidean time. The gravitational path integrals were introduced by Hawking and Gibbons [23] for the case of $U(1)$ symmetric spacetimes, and they gave the right result of the well known area law in rather general manner. LM further generalized these results with the use of conical excess and defects. One particular example of LM's prescription is the entanglement of subregion of a field theory with a gravity dual. In this case LM arguments explain the RT conjecture.
What we mean by a spacetime with $U(1)$ symmetry, is that the action of the theory is invariant under the transformation $\tau \to \tau + a$ for all $a \in \mathbb{R}$ for the Euclidean time τ. This symmetry implies a Killing vector ∂_τ which gives an exponential flow $exp(ia)$, and hence the name $U(1)$. One needs to realize that the full action is invariant under this transformation. This means that both the gravitational part and the matter fields coupled to the gravity need to respect this symmetry: $S_E[g, \phi] \to S_E[g, \phi]$.
The basis of the argument of Hawkings and Gibbons was to include the gravity itself in the path integrals and just like any other field introduce the appropriate boundary conditions to perform the integration:

$$\begin{aligned} Z &= \int Dg\, D\phi\, e^{-S_E[g,\phi]}, \\ S_E &= -\frac{1}{16\pi G_N} \int \sqrt{g}(R + ...) + (boundary\ terms) + (minimally\ coupled\ action\ of\ \phi), \end{aligned} \tag{5.1}$$

where ϕ can denote any set of matter fields. The next step in the calculation is to consider the periodic Euclidean time to be able to easily calculate the entropy using the thermodynamical methods. But the integration is neither easy nor yet well defined, so a saddle point approximation can be helpful:

$$Z(\beta) \approx exp(-S_E[\tilde{g}, \tilde{\phi}] + S^{(1)} + ...). \tag{5.2}$$

The leading term in the exponent is the action calculated with the classical solutions denoted by tilde. But the procedure is still not complete. Because of presence of a boundary at the given setup, one has to introduce the appropriate boundary term called the Gibbons-Hawking-York term into the action. Furthermore for action to be well-defined at large r on has to subtract divergent terms. At this point with the use of $U(1)$ symmetry and this boundary term the boundary divergences are subtracted using an appropriate gauge transformation and one can have a well defined semi-classical approximation of the partition function. Now all that remains is to use the familiar thermodynamical results to calculate the entropy which perfectly agrees with the area law. Doing the calculations with this method is not easy except for the simple cases like the Schwarzschild and the charged black holes. The $U(1)$ symmetry can be used to derive the area law in a much simpler manner with the use of conical defects. For that reason we only explain this method here. The reader who wishes to see the explicit calculations of the method of Hawking and Gibbons can consult to chapter 6 of [24].
But the $U(1)$ symmetry is not always present in the a general gravitational setup. And in fact the main purpose of LM's prescription is to generalize this result. LM assume a spacetime with a non-contractible circle in the boundary, which can be contracted in the bulk. using this method they develop a method to generalize the gravitational entropy to the Euclidean geometries without $U(1)$ symmetry. These solutions can be interpreted as calculating the trace of the density matrix of the full quantum gravity theory, in the classical approximation. The result of LM can be a proof for the general Ryu-Takayanagi formula [4] in the case of the Euclidean gravity.

5.1 Main idea of proof of Lewkowycz and Maldacena

As we said in the introduction, LM considers a boundary at which there is a direction that has the topology of the circle. Furthermore we assume that the boundary data depends on the position on this circle but it respects the periodicity. This means that we can define a coordinate $\tau \sim \tau + 2\pi$ on the boundary but we don't have a $U(1)$ symmetry for the coordinate τ since we assumed that position on this circle matters. We consider the interior spacetime to be smooth. This smoothness will correspond to a well defined energy tensor in the interior. This will mean that one can integrate the lagrangian of the theory over all points, even though there may be some defects in the geometry itself (we will see explicitly how this works out when we discuss the introduction of cosmic branes to derive the quantum extremal surfaces later). The circles in the boundary are assumed to be contractible in the interior. Roughly this means that if we shrink their radius coordinate they will reduce to a point in the interior. The euclidean action of the interior + the boundary is then denoted by $ln\, Z(1)$, where $Z(1)$ is the partition function for this setup.
The next step is to use the replica method. In order to do so, we patch n identical boundaries together so that we arrive at the same boundary with $\tau \sim \tau + 2\pi n$. This boundary by construction has the Z_n symmetry corresponding to the cyclic permutation of the copies. We assume that this symmetry is not spontaneously broken in the interior. This means that rotations by values of $2\pi k,\ k \in \{1, 2, ..., n\}$ in the τ coordinate would live the action invariant. This will be what we call the Z_n symmetric spacetime. The action for this setup is then given by $ln\, Z(n)$, where $Z(n)$ is the partition function of this replicated setup. Note that this setup is a more general case of the $U(1)$ symmetric case. And when we have the $U(1)$ symmetry, it is clear that these assumptions hold.
The un-normalized density matrix of a general quantum system is obtained by its Euclidean evo-

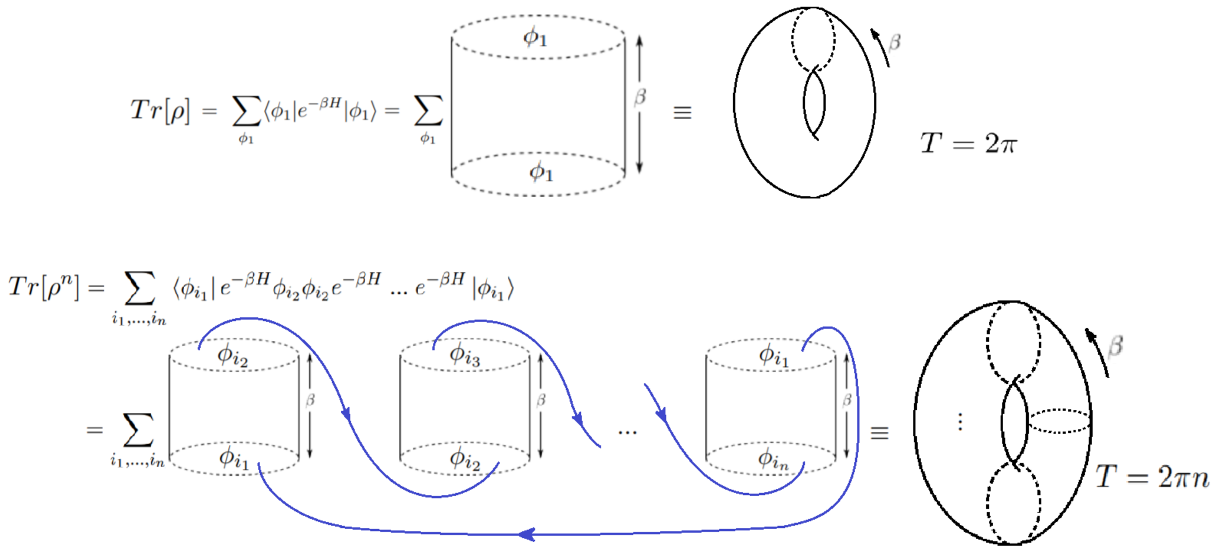


Figure 9: computation of $tr[\rho]$ and $tr[\rho^n]$ by path integral

lution as:

$$\rho = P e^{\int_0^{\tau_f} d\tau \, H(\tau)}, \tag{5.3}$$

where $H(\tau)$ is a time dependent Euclidean hamiltonian and P is the path-ordering operator. First we can compute $Tr[\rho]$ by considering Euclidean evolution on a circle. We set the the coordinate length of the initial circle to 2π, so $\tau_f = \tau_0 + 2\pi$. We can also compute $Tr[\rho^n]$ (n is integer) similarly by considering Euclidean evolution over a circle of n times the length of the original one (figure 9). The bulk evolution is periodic under shifts of the original circle like the boundary, $H(\tau + 2\pi) = H(\tau)$. The replica trick can then be applied to the setup:

$$\begin{gathered} S = -n\partial_n[\ln Z(n) - n \ln Z(1)]|_{n=1} = -n\partial_n[\ln(tr(\hat{\rho}^n)) - n\ln(tr(\hat{\rho}))]|_{n=1} = -tr[\hat{\rho} \ln \hat{\rho}], \\ Z(n) = tr[\rho^n], \qquad \hat{\rho} = \frac{\rho}{tr[\rho]} \end{gathered} \tag{5.4}$$

in which $\hat{\rho}$ is the normalized density matrix. We should note that first we need to calculate $Z(n)$ and then perform an analytic continuation in n.

Another implicit assumption is that gravity is holographic. we accept that setting boundary conditions on some boundary defines the theory and the interior (bulk) geometry is an approximation to the full computation (in low energy limit of string theory). The point is we only need it to be approximately valid to compute an approximately density matrix in some approximate theory. Everything get straight forward if the boundary is true asymptotic boundary. It corresponds to computing the entropy of a well defined density matrix in the dual field theory.

the final answer of the entropy is given by the area of a codimension-two surface in the bulk of original solution which is homologous to the boundary. being homologous means it shrinks smoothly to zero size. We will also see that the area must be minimal.

$$\boxed{S \equiv -n\partial_n[\ln Z(n) - n\ln Z(1)]|_{n=1} = \frac{A_{minimal}}{4G_N}} \tag{5.5}$$

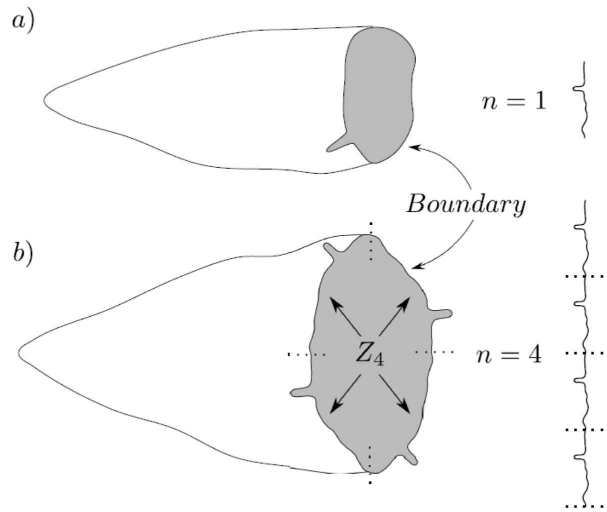


Figure 10: Gluing process of the boundaries. This figure is taken from [5].

To prove above equation in following sections we impose reasonable assumptions regarding the analytical continuation of the solutions away from the integer values of n. Einstein equation gives rise to the area term in 5.5 and we can use similar computation to generalize it to other actions.

5.2 Assumptions

To recap it can be good to remember what the assumptions are in the work of LM:

(1) Unlike the usual Gibbons-Hawking calculation the $U(1)$ symmetry is not needed. One only needs to consider a boundary with a circular direction, which cannot be contracted to a point in the boundary, very much like the closed strings wound around compact D-branes. This circle can be contracted in the bulk. This assumption also says that we are considering the Euclidean gravity.
(2) Different points on the circle do not have the same data. This means that the resulting replicated geometry only has a $\mathbb{Z}_n$ symmetry in the boundary. The next assumption is that this symmetry holds also in the bulk.
(3) Gravity is smooth, meaning one has a well defined stress energy tensor and it is possible to integrate the lagrangian over the whole setup.
(4) The holography is implicitly assumed for this general gravitational setup. Meaning that we are implicitly assuming that there exists a thermal field theory in the boundary. Of course except for the case of AdS gravity the exact nature of boundary is not known.

5.3 A simple example without $U(1)$ symmetry

Here we consider a simple example that LM brought up to show the main Idea's of their work. For this one can consider the Euclidean BTZ geometry:

$$ds^2 = \left[\frac{dr^2}{1+r^2} + r^2 d\tau^2 + (1+r^2)dx^2\right], \tag{5.6}$$

where one has $\tau \sim \tau + 2\pi$. x direction is an isometry of this metric and the functions we consider won't depend on it so we take it to be compact of size L_x. This metric is the classical gravitational metric and has $U(1)$ symmetry in τ direction, so the calculations due to Gibbons-Hawking argument, would give the usual area law formula S_0 for the entropy.
Next one considers a complex, minimally coupled, massless scalar ϕ, with the boundary conditions:

$$\phi = \eta e^{i\tau}, \quad at\ r \to \infty. \tag{5.7}$$

It is clear that if we transform the field with a global $U(1)$ transformation, these boundary conditions are not invariant, and this is where the $U(1)$ asymmetry of the setup is introduced. Now if we consider the gravitational action we have:

$$S_E[g,\phi] = S_E[g] + S_{scalar\ coupled\ to\ g}[\phi]. \tag{5.8}$$

It is apparent that any effect of non-zero order in η has to come from the second term, and in fact the first non-zero term has to be η^2 due to quadratic terms in the Lagrangian of the field ϕ. Now to use the replica trick one has to define the metric for the replicated geometry, which is defined to be:

$$ds^2 = \left[\frac{dr^2}{n^{-2}+r^2} + r^2 d\tau^2 + (n^{-2}+r^2)dx^2\right]. \tag{5.9}$$

In this spacetime to avoid singularity at $r = 0$ one has to impose $\tau \sim \tau + 2\pi n$. We can see this by expanding the metric near the horizon, $r \approx \epsilon$, $\epsilon \ll 1$:

$$ds^2 \approx n^2 dr^2 + \epsilon^2 d\tau^2 + ... \tag{5.10}$$

Now if we define $\epsilon = R/n$ we have:

$$ds^2 \approx dR^2 + R^2 d\alpha^2, \tag{5.11}$$

where $\alpha = \tau/n$. So to avoid singularity one needs, $\alpha \sim \alpha + 2\pi$ and thus $\tau \sim \tau + 2\pi n$.
Now one considers a scalar field in this spacetime. The equation of motion is the wave equation in the curved spacetime, and the solution which is regular at the origin is given by the hypergeometric functions:

$$\phi = \eta e^{i\tau} f_n(r), \quad f_n = (nr)^2 \frac{\Gamma(\frac{n}{2}+1)^2}{\Gamma(n+1)}\ {}_2F_1\left(\frac{n}{2}, \frac{n}{2}+1; n+1; -(nr)^2\right). \tag{5.12}$$

One has to note that $f_n \to 1$ as $r \to \infty$, so these solutions obey the suitable boundary condition 5.7 as it is needed. Now to use 5.5 one has to evaluate the gravitational action for each n. If one considers the corrections up to second order in η, using relations for the Euler's digamma function and hypergeometric function one has:

$$\begin{aligned}\ln\ Z(n)|_{\eta^2} &= -\int_{bulk}|\nabla\phi|^2 = -(2\pi n)L_x[r^3\phi^*\partial_r\phi]_{r=\infty} = \\ &= (2\pi L_x)[1 - n log\, n + n\psi(n/2) + (linear\ in\ n)],\end{aligned} \tag{5.13}$$

where as we said before L_x is the length of the x direction and ψ is the Euler ψ function or digamma function which is the first derivative of the gamma function. If one wants to calculate the action itself the linear terms in n has to be subtracted in order avoid the divergences, but in the calculation of entropy using the replica trick these terms will not contribute. The geometry of 5.9 is well defined if we replace n with non-integer values. So one can analytically continue in n and compute the entropy using 5.4:

$$\begin{aligned} S &= -n\partial_n[\ln Z(n) - n\ln Z(1)]|_{n=1} \\ &= S_0 + \eta^2 2\pi L_x\left(0 - 1 - 1 + \frac{1}{2}\frac{\pi^2}{2}\right) \\ &= S_0 + \eta^2\pi L_x(4 - \frac{\pi^2}{2}),\end{aligned} \tag{5.14}$$

where in the second equality we used the values of digamma function and trigamma function:$\psi(\frac{1}{2}) = 1, \psi_1(\frac{1}{2}) = \frac{\pi^2}{2}$. The calculation above gives the order η^2 correction to the area law, due to scalar field. If one computes the change using the Einstein's equation [5], the correction to the area of the horizon gives the same term. So this example explicitly checked the proposal by LM.

5.4 Regularization of Conical Defects

Before following the general arguments of Maldacena and Lewkowycz, it can be useful to review the work of Fursaev and Solodukhin on regularization of conical defects [25][26]. First we consider the cases where one has $U(1)$ symmetry in the geometry for some angular coordinate ϕ. Then we extend this to the case with discrete $\mathbb{Z}_n$ symmetry for ϕ which is the case that we are interested in, through our calculations. One has to note that by saying cone we mean any geometry with a conical singularity whether it has conical excess or conical deficit. We will talk about the meaning of these in more detail below.

5.4.1 $U(1)$ symmetric cones

We first look at cases with $U(1)$ symmetry in the angular coordinate ϕ, meaning for transformations $\phi \to \phi + x,\ \forall x \in \mathbb{R}$ the metric is invariant, and thus ∂_ϕ is a Killing vector. This can be readily seen from the independence of the metric components from the ϕ coordinate in the cases we mention in this section.

2-dimensional case: First it is instructive to look at the 2-dimensional cones which are the ones that we can easily visualize and do calculations with. First we consider a 2-dimensional cone with a conical deficit, meaning a geometry on which ϕ has a period less than 2π. This object can be viewed as a surface embedded in the 3-dimensional Euclidean space. One of the possible sets of coordinate functions for this surface are given by:

$$x = \alpha\rho\ \cos(\frac{\phi}{\alpha}), \quad y = \alpha\rho\ \sin(\frac{\phi}{\alpha}), \quad z = \sqrt{1-\alpha^2}\rho, \tag{5.15}$$

where $\phi \in [0, 2\pi\alpha)$, $1 \geq \alpha > 0$ and ρ is a positive real number. For $\alpha \neq 1$, this surface is regular for all points except for $\rho = 0$, and from the differential geometry one knows that it is not possible to define the tangent plane to this surface at $\rho = 0$, and this point is called a (conical) singularity at which the surface is not regular. For $\alpha = 1$ this surface is just a disk at $z = 0$ and is regular everywhere.

The equation for this surface using the coordinate functions 5.15 is given by:

$$z^2 - \frac{1-\alpha^2}{\alpha^2}(x^2+y^2) = 0\ ,\ \ z \geq 0. \tag{5.16}$$

This is the familiar equation of a conical surface in the 3-dimensional space, and if $\alpha \neq 1$ there is a singularity at $z = 0$, as we mentioned above. The induced metric on this surface can be found by pulling back the coordinate functions 5.15. The components of the pullback of the metric are $g_{\mu\nu}(x') = \sum_i \frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^i}{\partial x'^\nu}\eta_{ij}$, where x^i are the coordinate in the Euclidean space and x'^μ are coordinates on the submanifold which is our surface. So the non-zero components of the induced metric on the regularized cone are:

$$\begin{aligned} g_{\rho\rho} &= \alpha^2\cos^2(\phi/\alpha) + \alpha^2\sin^2(\phi/\alpha) + (1-\alpha^2) = 1 \\ g_{\phi\phi} &= \rho^2\sin^2(\phi/\alpha) + \rho^2\cos^2(\phi/\alpha) = \rho^2. \end{aligned} \tag{5.17}$$

Then the metric induced on this surface is given by:

$$ds_C^2 = d\rho^2 + \rho^2 d\phi^2, \tag{5.18}$$

where $\rho > 0$ and $\phi \in [0, 2\pi\alpha)$ with $1 \geq \alpha > 0$. What can we say if $\alpha > 1$? Introducing an embedded surface in Euclidean space for this case is not possible but one can argue using the metric itself. Let us imagine the same metric above with $\alpha > 1$. There is a well known result in Riemannian geometry that states that it is possible to construct a coordinate system on a small enough neighborhood of any point on a smooth manifold on which metric locally looks like the flat euclidean metric (Christoffel symbols vanish at this point). This coordinate is called the Riemann's normal coordinate. Now we can consider a constant ρ curve on this surface for small $\rho = \epsilon$. then one has for this curve:

$$\frac{perimeter}{radius} = \frac{2\pi\alpha\epsilon}{\epsilon} = 2\pi\alpha, \tag{5.19}$$

where the perimeter is just the length of this curve which is calculated by integrating the length element $\epsilon^2 d\phi^2$. The important point is that the expression above won't become 2π as $\epsilon \to 0$. It means that for any neighborhood no matter how small around $\rho = 0$ the geometry won't look flat. This means that the geometry cannot be extended smoothly to $\rho = 0$, because otherwise we would have been able to construct a locally flat coordinate around $\rho = 0$ (Riemann's normal coordinates can be defined around any point of a smooth manifold). The argument above shows a singular behavior for the case of $\alpha > 1$. Thus for both cases $1 > \alpha > 0$ and $\alpha > 1$ one has a singular

behavior. For the first case we say we have a conical singularity with conical deficit $2\pi(1-\alpha)$ and for the second case we say we have a conical singularity with a conical excess of $2\pi(\alpha-1)$.
Now we consider a two-dimensional space $\mathcal{M}_\alpha$ with the topology of a cone $\mathcal{C}_\alpha$. The metric on this space is defined to be that of a cone up to a conformal factor:

$$ds^2 = e^\sigma(d\rho^2 + \rho^2 d\phi^2) := e^\sigma\, ds^2_C, \tag{5.20}$$

where ds^2_C is the line element on $\mathcal{C}_\alpha$, and ϕ runs from 0 to $2\pi\alpha$, but note that by arguments above we have $\alpha > 0$, meaning we consider both the conical deficits and conical excesses. The conformal factor σ is assumed to have the following expansion in the vicinity of $\rho = 0$:

$$\sigma = \sigma_1\rho^2 + \sigma_2\rho^4 + \dots\,. \tag{5.21}$$

The functions σ_1 and σ_2 are in general function of the angle ϕ and a possible constant term which can be absorbed by redefinition of ρ. This conformal factor allows us to consider even more general geometries than the usual cones we introduced in detail above. This behavior of the conformal factor ensures us that the singularity at $\rho = 0$ can only be due to the conical metric ds^2_C, in other words the conformal factor should not be singular at $\rho = 0$.
Now our aim is to mend this geometry at $\rho = 0$ to make it regular at this point as well. Let us go back to the case for $1 > \alpha > 0$. Then one can introduce a regularized cone $\tilde{\mathcal{C}}_\alpha$ with equation $z = \sqrt{|1-\alpha^2|}f(\rho, a)$ for the z coordinate, where $f(\rho, a)$ is a smooth function and a is a regularization parameter such that $lim_{a\to 0} f = \rho$.
Furthermore this regularized coordinate must have a minimum at $\rho = 0$ just like the original z in $\mathcal{C}_\alpha$. This additional condition implies $\partial_\rho f\big|_{\rho=0} = 0$. We also need that away from $\rho = 0$, the geometry coincides with the usual cone, so we need $\partial_\rho f\big|_{\rho\gg 1} = 1$. To find the metric in the regularized cone we remember the embedding, $x = \alpha\rho\cos(\phi/\alpha)$, $y = \alpha\rho\sin(\phi/\alpha)$, $z = \sqrt{1-\alpha^2}f(\rho, a)$ and pull the metric back to the two-dimensional submanifold of the Euclidean space. Again the non-zero components of the induced metric on the regularized cone are:

$$\begin{aligned}
g_{\rho\rho} &= \alpha^2\, \cos^2(\phi/\alpha) + \alpha^2\, \sin^2(\phi/\alpha) + (1-\alpha^2)(\partial_\rho f)^2 = \alpha^2 + (1-\alpha^2)(\partial_\rho f)^2\\
g_{\phi\phi} &= \rho^2\, \sin^2(\phi/\alpha) + \rho^2\, \cos^2(\phi/\alpha) = \rho^2.
\end{aligned} \tag{5.22}$$

With this argument for $1 > \alpha > 0$ in mind, we define the metric for the regularized cone $\tilde{\mathcal{C}}_\alpha$ for all $\alpha > 0$ by

$$ds^2_{\tilde{C}} = u d\rho^2 + \rho^2 d\phi^2, \quad u\big|_{\rho=0} = \alpha^2, \quad u\big|_{\rho\gg a} = 1. \tag{5.23}$$

u can be any function with the desired asymptotic behavior. For example for the case of $1 > \alpha > 0$ above we have:

$$u = \alpha^2 + (1-\alpha^2)(\partial_\rho f)^2, \tag{5.24}$$

for which the asymptotic behavior is consistent due to the properties of f.
Now we return to the case of the original manifold 5.20, and we replace the singular cone with the regularized one. We call this space $\tilde{\mathcal{M}}_\alpha$. Conformally transformed Ricci scalar can be calculated readily:

$$R = e^{-\sigma} R_{\tilde{\mathcal{C}}} - e^{-\sigma}\nabla_\mu\nabla^\mu\sigma, \tag{5.25}$$

where $R_{\tilde{\mathcal{C}}}$ is the Ricci scalar for the regularized cone and ∇ is the covariant derivative with respect to the regularized cone's metric, $ds^2_{\tilde{\mathcal{C}}}$. We have to note that the volume element in the $\tilde{\mathcal{M}}_\alpha$ is

$d\mu = e^{\sigma}\sqrt{u}\rho d\rho d\phi$. Using this volume we can calculate the following integral:

$$
\begin{aligned}
\int_{\tilde{\mathcal{M}}_\alpha} R &= \int_0^{2\pi\alpha}\int_0^{\infty} e^{\sigma}u^{1/2}\rho d\rho d\phi\, e^{-\sigma}R_{\tilde{\mathcal{C}}} - \int_0^{2\pi\alpha}\int_0^{\infty} e^{\sigma}u^{1/2}\rho d\rho d\phi\, e^{-\sigma}\nabla_\mu\nabla^\mu\sigma \\
&= 2\pi\alpha\int_0^{\infty} d\rho\,(\partial_\rho u)u^{-\frac{3}{2}} - \int_0^{2\pi\alpha}\int_0^{\infty} u^{1/2}\rho d\rho d\phi\,\nabla_\mu\nabla^\mu\sigma \\
&= 2\pi\alpha(-2u^{-1/2})\big|_0^{\infty} - \int_0^{2\pi\alpha}\int_0^{\infty} u^{1/2}\rho d\rho d\phi\,\nabla_\mu\nabla^\mu\sigma \\
&= 4\pi(1-\alpha) - \int_0^{2\pi\alpha}\int_0^{\infty} u^{1/2}\rho d\rho d\phi\,\nabla_\mu\nabla^\mu\sigma,
\end{aligned}
\tag{5.26}
$$

where in the second line we used the Ricci scalar formula for the regularized cone, and in the last line we used the asymptotic behavior of u function, 5.23. as one can see, the first term in the above integral is independent of the regularization. The dependence on u appears only in the second term.
Let us denote the conical singularity by Σ which in the present case is a single point $\rho = 0$ set, but later in the higher dimensional cases it can be a nontrivial set. This set is the fixed point of the action of $U(1)$ since it is normal to the ϕ coordinate. Now imagine we consider the original $\mathcal{M}_\alpha$ and omit Σ from it. Then everywhere on $\mathcal{M}_\alpha\backslash\Sigma$ we have

$$
R = e^{-\sigma}(R_{\mathcal{C}} - \nabla^\mu\nabla_\mu\sigma) = -e^{-\sigma}(\nabla^\mu\nabla_\mu\sigma), \tag{5.27}
$$

where we used the fact that Ricci scalar vanishes everywhere on $\mathcal{C}_\alpha$ except for the tip. Then using the expression above in 5.26 in the limit $a \to 0$ we have:

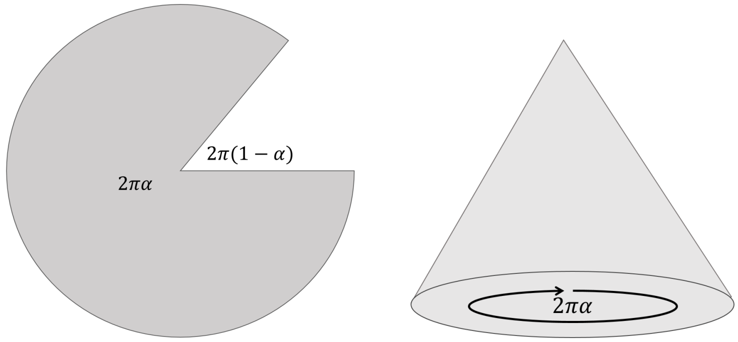
$$
\begin{aligned}
\lim_{\tilde{\mathcal{M}}_\alpha\to\mathcal{M}_\alpha}\int_{\tilde{\mathcal{M}}_\alpha} R &= 4\pi(1-\alpha) - \int_0^{2\pi\alpha}\int_0^{\infty} e^{\sigma}u^{1/2}\rho d\rho d\phi\,(e^{-\sigma}\nabla_\mu\nabla^\mu\sigma) \\
&= 4\pi(1-\alpha) + \int_{\mathcal{M}_\alpha\backslash\Sigma} R,
\end{aligned}
\tag{5.28}
$$

where we used 5.27 and the volume element of $\mathcal{M}_\alpha$. Here again we remember that $\mathcal{M}_\alpha\backslash\Sigma$ is the cone with the singularity removed from its tip. This result and its generalization to higher dimensional cases will be important for us.

Higher-dimensional case: In this section we generalize the result in the previous section, but we won't do the calculations in detail. For the full details of this calculation one can refer to [25]. Here we consider a two-dimensional cone $\mathcal{C}_\alpha$ in the Riemannian d-dimensional manifold $\mathcal{M}_\alpha$ such that near the singularity $\rho = 0$ the metric is represented as

$$
ds^2 = e^{\sigma}(d\rho^2 + \rho^2 d\phi^2 + \sum_{i,j=1}^{d-2}(\gamma_{ij}(\theta) + h_ij(\theta)\rho^2)d\theta^i d\theta^j + \ldots) := e^{\sigma}d\tilde{s}^2, \tag{5.29}
$$

where ... means terms of higher power in ρ^2. ϕ runs from 0 to $2\pi\alpha$ for all $\alpha > 0$. We used the same parametrization as in two dimensions but, here the singular set is a $(d-2)-$dimensional surface Σ with coordinates $\{\theta^i\}$ and the metric $\gamma_{ij}(\theta)$.



(a) (b)

Figure 11: (a) An expanded cone with the deficit angle $2\pi(1-\alpha)$. (b) The same cone after the identification. The circumference of the circle at the bottom is $2\pi\alpha$ as expected.

Following the same route of the last section we define the metric for the regularized space $\tilde{\mathcal{M}}_\alpha$ with a parameter a by changing the $g_{\rho\rho}$ component in the conical part of the metric

$$ds^2 = e^\sigma (u(\rho, a)d\rho^2 + \rho^2 d\phi^2 + \sum_{i,j=1}^{d-2} (\gamma_{ij}(\theta) + h_i j(\theta)\rho^2) d\theta^i d\theta^j + ...). \tag{5.30}$$

again for u we have the following behavior:

$$u\Big|_{\rho=0} = \alpha^2, \quad u\Big|_{\rho\gg a} = 1. \tag{5.31}$$

Now in 5.30 again we have $g_{\mu\nu} = e^\sigma \tilde{g}_{\mu\nu}$ and remember the Weyl transformed Ricci scalar:

$$R[g] = e^{-\sigma}\bigg(R[\tilde{g}] - \tilde{\nabla}^\mu \tilde{\nabla}_\mu \sigma\bigg). \tag{5.32}$$

Now follwing a calculation similar to the that of 2-dimensional case we have:

$$\boxed{\lim_{\tilde{\mathcal{M}}_\alpha \to \mathcal{M}_\alpha} \int_{\tilde{\mathcal{M}}_\alpha} R = 4\pi |1-\alpha| A_\Sigma + \int_{\mathcal{M}_\alpha \backslash \Sigma} R,} \tag{5.33}$$

where A_Σ is the area of the codimension-two of the conical singularity, which is the fixed point of $U(1)$.

5.4.2 Squashed cones

The analysis of the last section was limited to the case of conical singularities in spaces with $U(1)$ symmetry. In a more general case there will not be $U(1)$ isometry in the metric for which Σ is a fixed point set. One can instead consider a conical space which has a discrete group of transformations $\phi \to \phi + 2\pi k, \;\; k \in \{1, 2, ..., n\}$. These kinds of cones are referred to as squashed cones.
It is good to have a specific metric to describe these geometries, and as we shall see this form of

metric will play a role in the argument of LM. From now on we call the angular coordinate τ instead of ϕ to have a more physical notation. The general form of the metric in a static spacetime is

$$ds^2 = g_{tt}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j. \tag{5.34}$$

We consider Riemannian manifolds in which we have $g_{tt}(x^k) > 0$. We can choose a codimension=two surface Σ in this space in a constant time slice. The aim is to understand the properties of this surface and generalize them to the case of a spacetime which is not static. Around Σ in the constant time slice, one can construct the Riemann normal coordinates and write the inner product:

$$h_{ab}(x)dx^a dx^b = dr^2 + (\gamma_{ij}(y) + 2rK_{ij}(y) + O(r^2))dy^i dy^j, \tag{5.35}$$

where r is the geodesic distance from a point in the time slice to Σ and y^i and γ_{ij} are the coordinates and the metric on Σ. K_{ij} is also the extrinsic curvature of Σ. Here h_{ij} is a tensor that determines the induced inner product on the constant time slice. Then by introducing another coordinate $\zeta = \sqrt{g_{tt}}t$, we have

$$d\zeta = \sqrt{g_{tt}}dt + \zeta w_a dx^a, \tag{5.36}$$

where $w_a = \frac{1}{2}\partial_a g_{tt}/g_{tt}$. Now from 5.34,5.35, near Σ we have:

$$ds^2 \simeq d\zeta^2 + dr^2 + (\gamma_{ij}(y) + 2rK_{ij}(y))dy^i dy^j - 2\zeta w_r(y)d\zeta dr - 2\zeta d\zeta w_i(y)dy^i. \tag{5.37}$$

One can make an additional coordinate transformation

$$v^i = y^i - \frac{1}{2}\zeta^2 w^i(y), \quad \bar{r} = r - \frac{1}{2}\zeta^2 w_r(y). \tag{5.38}$$

to arrive at:

$$ds^2 \simeq d(x^1)^2 + d(x^2)^2 + (\gamma_{ij}(v) + 2x^2 K_{ij}(v))dv^i dv^j, \tag{5.39}$$

where we have $x^1 = \zeta$ and $x^2 = \bar{\rho}$ and omitted the terms of quadratic order. In the case of a static spacetime Σ has a single non-vanishing extrinsic curvature. The generalization of 5.39 to the case with two non-trivial curvatures is straightforward:

$$ds^2 \simeq d(x^1)^2 + d(x^2)^2 + (\gamma_{ij}(v) + 2\sum_{p=1,2} x^p K^p_{ij}(v))dv^i dv^j, \tag{5.40}$$

where now K^p_{ij} are extrinsic curvatures of Σ for normals n_p. Another transformation with $x^1 = r\sin\tau, x^2 = r\cos\tau$ brings us to:

$$ds^2 \simeq r^2 d\tau^2 + dr^2 + (\gamma_{ij}(v) + 2r\cos\tau K^1_{ij}(v) + 2r\sin\tau K^2_{ij}(v))dv^i dv^j, \tag{5.41}$$

These kinds of metrics characterize the general behavior that one expects for a squashed cone. Now again if we have for τ coordinate $\tau \in [0, 2\pi\alpha)$ for $\alpha > 0$ and $\alpha \neq 1$, again we have a general space with conical excess of deficit. Now if $\alpha = n$ is an integer, from 5.41 we can see that we have a symmetry $\tau \to \tau + 2\pi k,\ k \in \{1, 2, ..., n\}$ which is the statement that this geometry is $\mathbb{Z}_n$ (replica) symmetric.

Fursaev and Solodukhin further showed [26] that in this case again we can write

$$\boxed{\lim_{\tilde{\mathcal{M}}_n \to \mathcal{M}_n} \int_{\tilde{\mathcal{M}}_n} R = 4\pi|1-n|A_\Sigma + \int_{\mathcal{M}_n \backslash \Sigma} R + ...,} \tag{5.42}$$

where in here again $A(\Sigma)$ is the area of the fixed point set of $\mathbb{Z}_n$ which is the singular codimension-two surface. The ... are $O(|n-1|^2)$ corrections which are regulation dependent which are absent in the case where the dimension of the full spacetime is 2. This result will be the regularization that is needed for the LM prescription.

5.5 Computation of the Gravitational Entropy when there is a $U(1)$ symmetry

Here we explain the two methods introduced in [5] for calculating the entropy when there is a $U(1)$ symmetry to re-derive the results of Hawking and Gibbons.

5.5.1 Entropy From Rounded off Cones

Our setup has a circle in the boundary furnished with the coordinate τ. One sets the period of the circle to be $\tau \sim \tau + 2\pi n$ and then with the use of the replica method one has:

$$S = -n\frac{\partial}{\partial n}\ln(Z(n)/(Z(1))^n)_{n=1} = -n\partial_n[\ln Z(n) - n\ \ln Z(1)]_{n=1}. \tag{5.43}$$

The first term in the brackets above is the smooth action when $n > 1$. This is an important fact to understand, because this means that not every τ coordinate with the $2\pi n$ period is a conic space. The first term above is a geometry made by patching n copies of the original and is smooth by construction since there is no excess angle. We just have n copies. the second term can be interpreted as the solution for $n = 1$ but with τ identified so that $\tau \sim \tau + 2\pi n$. This is where we have a conical excess, because in the non-singular metric for $n = 1$, increasing the period of τ results in a conical excess. This interpretation is possible because we have a $U(1)$ translational symmetry in $\tau \to \tau + a$, which is the symmetry of theory by assumption, then we can write:

$$n \ln Z(1) = n \int_0^{2\pi} d\tau \mathcal{L}_1 = \int_0^{2\pi n} d\tau \mathcal{L}_1. \tag{5.44}$$

So the calculation above says that one can view the second term in 5.43 as calculating the action of a geometry with a conical excess, of strength $2\pi(n-1)$. Next step in the calculation is not to include the contributions from the conical singularity, or more precisely regulate it with the methods we introduced in the last section. To do this one integrates the gravitational action density away from the tip which can be seen in Figure 12.
As a next step one adds and subtracts rounded off cones to the 5.43, in order to make the calculations easier. These rounded of cones are the regulated cones constructed in the regularization of the $U(1)$ symmetric cones in the last section. One has to remember that these cones are not solutions of the equations of the motion. They are just a mere tool for calculation. These cones are chosen so that they agree with the singular geometry far from the tip, (at $r > a$ in Figure 12) but they are rounded at the tip. furthermore it is possible to choose them in a way that they differ only by an amount of order $n - 1$ from the true solution (see eq. 5.33) which is the ln $Z(n)$. With these configurations the expression 5.43 becomes:

$$S = -n\partial_n\left[(\ln Z(n) - \ln Z^{off}(n)) - (\ln Z^{off}(n) - n \ln Z(1))\right]_{n=1}. \tag{5.45}$$

Then one interprets the first parenthesis as a first order variation by a continuous parameter n from the original solutions (remember that we chose the off-shell configuration of rounded off cones so that they differ only by an amount of order $n - 1$). because the first variation of the action is to vanish when one considers the equations of motion, then the first parenthesis above vanishes, by the definition of the first variation. For the second parenthesis the contribution comes only from

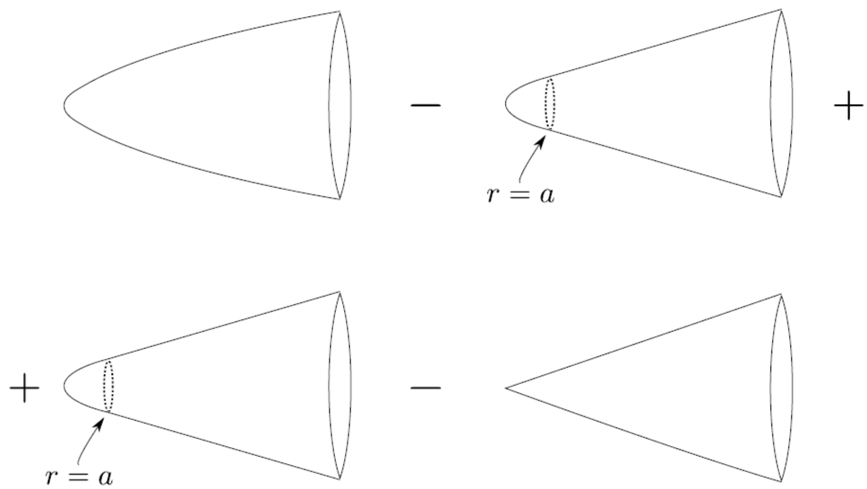


Figure 12: The first geometry is the right solution with period $2\pi n$ which gives $Z(n)$. There is a conical singularity in the last geometry. It is the solution of $n = 1$ after the circle identified periodic over $2\pi n$ ($\tau \to \tau + 2\pi n$) which gives $nZ(1)$. The deficit angle of the cone almost vanishes in $n = 1$ and it's been exaggerated in the picture. We subtract and add the middle cone corresponding to the regularized version of the $n = 1$ solution. It's geometry is the same as the last one for $r > a$, where a is small regulator. Obviously it's not a solution and is just an off-shell configuration. All of the above configurations obey the same boundary conditions at infinity. This figure is taken from [5].

the tip of the cones. This is due to the fact that we chose the off-shell cones so that they only differ near the tip from the singular cone, meaning that we use 5.33. This contribution near the tip using 5.33 can be written as an integral:

$$\int_{Reg\ cone} d^2x\sqrt{g}R \sim 4\pi(1-n), \tag{5.46}$$

where x are coordinates along the cone. Then one can readily find the area law formula using 5.45 and plugging in the gravitational action:

$$S = \frac{1}{16\pi G_N} A_\Sigma \bigg(-n\partial_n \int_{Reg\ cone} d^2x\sqrt{g}R \bigg) = \frac{A_\Sigma}{4G_N}, \tag{5.47}$$

where again Σ is the codimension-two surface corresponding to the fixed points of $U(1)$. Of course this does not show that Σ is minimal. To show that one needs further arguments which will provide when discussing the general case involving the $\mathbb{Z}_n$ symmetric generalization of this result.

5.5.2 Apparent Conical Singularities

Another way of finding the entropy formula for the case of $U(1)$ symmetry is to use the $U(1)$ symmetry this time on the replicated geometry:

$$\ln\ Z(n) = \int_0^{2\pi n} d\tau\, \mathcal{L}_n = n\int_0^{2\pi} d\tau\, \mathcal{L}_n = n[\ln\ Z(n)]_{2\pi}. \tag{5.48}$$

Here $[ln\, Z(n)]_{2\pi}$ is the gravitational density for the solution labeled by n but integrated over τ from $[0, 2\pi]$ instead of $[0, 2\pi n]$. Now we can calculate the entorpy:

$$\begin{aligned} S &= -n\partial_n[\ln\ Z(n) - n\ln\ Z(1)]_{n=1} \\ &= -n\partial_n[n[\ln\ Z(n)]_{2\pi} - n\ln\ Z(1)]_{n=1} \\ &= -(\ln\ Z(1) - \ln\ Z(1)) - n^2\partial_n[\ln\ Z(n)]_{2\pi}\Big|_{n=1} = -n^2\partial_n[\ln\ Z(n)]_{2\pi}\Big|_{n=1}. \end{aligned} \tag{5.49}$$

In this case if we want to view the configuration with the period 2π. So we have a conical singularity with the opening angle of $2\pi/n$ or the deficit angle of $2(n-1)\pi/n$. For example if we have at $r \to 0$, like 5.11, $ds^2 \approx n^2dr^2 + \epsilon^2 d\tau^2 + ...$ Then $\tau \sim \tau + 2\pi$ corresponds to a conical singularity with the deficit angle $2\pi(n-1)/n$.

To calculate the entropy we do not include the curvature effects from the conical singularity, meaning we effectively setup a boundary near $r = 0$ and consider it with all other boundaries of the theory. This accounts to adding a cosmic brane at the position of the singularity so that the total action is smooth even though the geometry is singular. We will discuss this cosmic branes more thoroughly when we discuss the quantum extremal prescription in the later chapters. Since the variation of fields vanish due to the equations of motion, the only contribution in varying n comes from the boundary terms:

$$\begin{aligned} -\partial_n[\ln\ Z(n)]_{2\pi}\Big|_{n=1} &= \int (E_g\partial_n g + E_\phi \partial_n\phi) \\ &\quad + \frac{1}{8G_N}\int_{r\sim 0} d^{N-2}y\sqrt{g}(\nabla^\mu\partial_n g_{\mu r} - g^{\mu\nu}\nabla_r\partial_n g_{\mu\nu}), \end{aligned} \tag{5.50}$$

where E_g and E_ϕ are the equations of motion for the metric and other fields and N is the spacetime dimension. In the calculation above we realized that the only boundary term that contributes is due to the boundary that we imposed near $r \sim 0$, because we assumed that variations of the fields vanish at other boundaries. Furthermore we assumed that equations of motion hold and thus the first two terms above vanish as well. in the last integral above y are coordinates along the codimension-two surface near $r = 0$. A suitable parameterization for the two directions normal to this surface is $ds^2 \approx n^2dr^2 + r^2d\tau^2$, which we are familiar with, in the context of conical singularities like the example we considered in previous sections. Now it is clear that only components of metric that contribute to the derivatives are those that include n, namely the only non-vanishing term is $\partial_n g_{rr}|_{n=1} = 2$. considering these the term in the parenthesis in 5.50 gives $2/r$ and we have:

$$\begin{aligned} -\partial_n[\ln\ Z(n)]_{2\pi}\Big|_{n=1} &= \frac{1}{8G_N}\int_{r\sim 0} d^{N-2}y\sqrt{g}(\nabla^\mu\partial_n g_{\mu r} - g^{\mu\nu}\nabla_r\partial_n g_{\mu\nu}) \\ &= \frac{1}{8G_N}\int_{r\sim 0} d^{N-2}y\,\sqrt{\gamma}\left(\sqrt{r^2n^2}(\frac{2}{r})\right) = \frac{A_\Sigma}{4G_N}, \end{aligned} \tag{5.51}$$

where A_Σ is the area of the codimension two surface which we parameterized by y coordinates and γ_{ij} is the induced metric on this surface. This is the area law derived from another route and again the minimality condition is to be proved.

5.5.3 An Example

Before proceeding forward, it is instructive to see how this method gives the entropy for an actual example. We look at the Schwarzschild black hole:

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{5.52}$$

To construct the euclidean version we pinch of the geometry at $R = 2MG$ and change the signature:

$$ds^2 = \left(1 - \frac{2MG}{r}\right)d\tau^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \;\; r > 2MG. \tag{5.53}$$

Now the fixed point of $\tau + a, a \in \mathbb{R}$ transformation is sphere at $r = 2MG$. Now using the formula derived above we have:

$$S = \frac{A}{4G} = \frac{4\pi(2MG)^2}{4G} = 4\pi M^2 G, \tag{5.54}$$

Which is the known formula for the Schwarzschild black hole.

5.6 Arguments to Generalize the Entropy Formula to the $\mathbb{Z}_n$ Symmetric Case

Now it is time to Generalize the results of the previous section to the case where we do not have the $U(1)$ symmetry on the direction of the circle. Now this is replaced by a weaker $\mathbb{Z}_n$ symmetry. The first thing to derive is to show that entropy does indeed correspond to the area of a codimension-two surface. This is not hard to show given the things we demonstrated so far.

We expect that our original solution for $n = 1$ is only $\mathbb{Z}_n$ symmetric for the integer n. Now the eq. 5.44 still holds and by writing $n \ln Z(1)$ we arrive at a geometry with $\mathbb{Z}_n$ symmetry which has a conical excess of $2\pi(n-1)$, namely a squashed cone. We can construct rounded of cones for this case as well so with the use of equations of motion we again arrive at:

$$S = n\partial_n \big[\ln Z^{off} - n \ln Z(1)\big]_{n=1}. \tag{5.55}$$

Now since we regulated a squashed cone with no $U(1)$ symmetry, we use 5.42 and arrive at:

$$\begin{aligned} S &= -\frac{1}{16\pi G_N}(A_\Sigma)\left(n\partial_n \int_{Reg\ cone} d^2x\sqrt{g}R\right)\Bigg|_{n=1} + ... \\ &= -\frac{1}{16\pi G_N}(A_\Sigma)\Big(n\partial_n(4(1-n)\pi)\Big)\Bigg|_{n=1} + ... \\ &\Longrightarrow \boxed{S = \frac{A_\Sigma}{4G_N} + ...\,,} \end{aligned} \tag{5.56}$$

where ... are the $O(|1-n|^2)$ terms, and A_Σ is the area of the codimension-two fixed point set of $\mathbb{Z}_n$. We performed the differentiation because the final result of subtracting two cones could be easily considered as a function of the real parameter n. Now that we found the area law all that remains is to find the minimality condition of Σ. To that aim we first discuss the meaning of the analytic continuation more clearly for the metric and fields.

5.6.1 Metric and Fields for Integer n

First we study the case of integer n. The boundary has the topology of a circle. We label the coordinate that runs over the circle by τ. Furthermore we are considering the cases in which the circle is non-contractible on the boundary while being contractible in the interior. Boundary can be an asymptotic boundary like that of the AdS or any other surface on which one imposes boundary conditions. Moreover due to periodicity of the coordinate τ, the metric and all fields in the configuration are periodic on this circle. We can denote the set of all these fields with $\psi(\tau)$, such that:

$$\psi(\tau) \sim \psi(\tau + 2\pi). \tag{5.57}$$

One has to note the above notation does not exclude the dependence of these fields on other coordinates, and they can in general depend on other coordinates as well, even though we are only showing the τ dependence. The boundary conditions for the fields must also be periodic:

$$\psi(\tau)\big|_{Boundary} = \psi_B(\tau), \quad \psi_B(\tau) = \psi_B(\tau + 2\pi). \tag{5.58}$$

The replicated solution for $n > 1$ is constructed by patching together n identical copies of the boundary. So the coordinate τ now goes from 0 to $2\pi n$, but by construction the periodicity of the fields in the boundary still obeys the 5.58. This means that by construction the solution for $n > 1$ is replica symmetric, meaning it is invariant under rotations of $2\pi k,\ k \in \{1, 2, ..., n\}$ in the τ coordinates. We assume that this $\mathbb{Z}_n$ symmetry continues to hold in the bulk, or more precisely it does not spontaneously break in bulk.
For each of the solutions for $n > 1$ we assume that there exists a special codimension-two surface which is left invariant by the action of $\mathbb{Z}_n$. For two dimensions normal to this surface we assign a radial coordinate r which is a radial coordinate away from this surface and an angle τ which is the coordinate on the circle. As we seem before near this surface the metric in two dimensions transverse to this surface has the form:

$$ds^2 = n^2 dr^2 + r^2 d\tau^2 + \dots, \tag{5.59}$$

where the factor of n comes from demanding that there is no singularity at $r = 0$. This demand is justified because the nonsingular period for the metric constructed from n replicas is $2\pi n$ and there is no excess angle. We saw this kind of metric in the example without $U(1)$ which we analyzed before.
We said that we assume that there exists a fixed surface under the $\mathbb{Z}_n$ action in our setup. In fact this is justified by our very first assumption in regard to the existence of contractible circles in the bulk. Choosing the coordinate τ in a way that do not allow a fixed point for the $\mathbb{Z}_n$ action means that the circles won't shrink to a point and thus our assumption of existence of such point is natural.

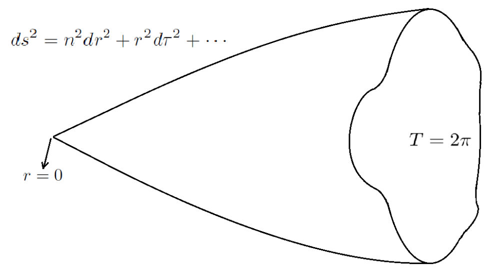


Figure 13: configuration introduced for non-integer n

5.6.2 Metric and Fields for non-Integer n

For the case for non-integer n we continue to impose exactly the same boundary condition 5.58 with period 2π. Obviously this cannot be compatible with $\tau \to \tau + 2\pi n$ when n is not an integer. When we have circle with positive size (non-zero circumference) we integrate from 0 to 2π and multiply the result by n. However as we mentioned before we expect to have surface on which this circle shrinks to zero radius. In two transverse directions to this codimension-two surface we impose that the metric continues to behave like 5.59 even for non-integer n. All of other fields including the other metric components which are for the coordinates on the surface are chosen to be periodic in $\tau \to \tau + 2\pi$ like 5.57. This will cause singularity at $r = 0$ for the field configuration, but we assume that this singularity is harmless like the example we introduced earlier.
The procedure above can be done in a more systematic way. One can compactify the τ direction to arrive at a conical singularity with the opening angle of $2\pi/n$. Then integrate the action without considering the contributions from the tip of the cone and then multiply it by n. This will give the same result as the one in the last paragraph. This procedure of compactifying and omitting the tip may seem a bit ad hoc at the moment but it can be explained rather clearly by introduction of a cosmic brane at the position of the tip of the cone. We will explain this when we discuss the derivation of the quantum extremal surface condition later on.
As $n \to 1$ the whole configuration goes over to the case with $n = 1$. Thus, we expect that the analytically continues solution is close to the case of $n = 1$ for small $n - 1$ and we can expand it in powers of $n - 1$.

5.6.3 Derivation of the Minimal Surface Condition

One can start from a simple situation of two-dimensional dilaton gravity which is given by the action:

$$-S = \frac{1}{16\pi} \int d^2x \sqrt{g} e^{-2\phi} [R + 4(\nabla\phi)^2 + ...], \tag{5.60}$$

where ϕ is the dilaton field, and the dots indicate other fields. We demand that the equations motion hold up to linear terms in $\epsilon = n - 1$ near $r = 0$. We will find that, expanding the fields around $n = 1$ solution, and assuming the periodicity condition for the fields will give $\partial_i \phi = 0$ if the equations of motion are satisfied.
for $n = 1$ we assume that we have some coordinate system x^1, x^2 and the metric is of the form

$ds^2 = d(x^1)^2 + d(x^2)^2 + O(x^2)$. For the case $n-1=\epsilon$ we expect a metric of the form $ds^2 = e^{2\rho}(d(x^1)^2 + d(x^2)^2)$, with $e^{2\rho} = r^{2(\frac{1}{n}-1)}$, as $r \to 0$. To the first order in ϵ we have:

$$\rho = (\frac{1}{n} - 1)\ln(r) \sim -\frac{1}{n^2}(\epsilon)\ln(r)\big|_{n=1} = -\epsilon \ln(r). \tag{5.61}$$

The equations of the motion for the action above are:

$$\begin{aligned} 0 &= e^{-2\phi}(4\partial_z\phi\partial_z\rho + 2\partial_z^2\phi) + T_{zz}^{matter} \\ 0 &= e^{-2\phi}(4\partial_{\bar z}\phi\partial_{\bar z}\rho + 2\partial_{\bar z}2\phi) + T_{\bar z\bar z}^{matter}, \end{aligned} \tag{5.62}$$

where $z = x^1 + ix^2, \bar z = x^1 - ix^2$, and T^{matter} is the stress tensor due to the dots in the action. Now if we expand to the first order in ϵ, using 5.61 we have:

$$\partial_z\rho = \frac{\partial}{\partial z}\rho \simeq \frac{\partial}{\partial z}(-\epsilon\ln(r)) = \frac{\partial}{\partial z}(-\epsilon\ln(\sqrt{z\bar z})) = -\epsilon\frac{1}{\sqrt{z\bar z}}\frac{\bar z}{2\sqrt{z\bar z}} = -\epsilon\frac{1}{2z}. \tag{5.63}$$

Now if we plug the result above into the first equation of motion we have up to first order in ϵ near $r=0$:

$$-2\partial_z\phi(0)\frac{\epsilon}{z} + 2\partial_z^2\delta\phi + \delta T_{zz}^{matter} = 0, \tag{5.64}$$

and a similar equation is found by expanding the second equation. $\partial_z\phi(0)$ is the derivative of the field for $n=1$ at the origin and is just a z independent constant. We want the singularity of the fields to be harmless at $r=0$ so we need that the stress tensor be non-singular at order $1/r \sim 1/z$, and thus we have:

$$\partial_z^2\delta\phi \propto \frac{\epsilon\partial_z\phi(0)}{z}, \quad \partial_{\tilde z}^2\delta\phi \propto \frac{\epsilon\partial_{\tilde z}\phi(0)}{\tilde z}. \tag{5.65}$$

The periodicity condition implies that if one takes the τ derivative of any field and integrate over τ between 0 and 2π, one should get zero. Now one can consider the following combination:

$$\partial_\tau[(r\partial_r - 1)\partial_z\delta\phi] \propto (z\partial_z - \tilde z\partial_{\tilde z})(z\partial_z + \tilde z\partial_{\tilde z} - 1)\partial_z\delta\phi \propto \epsilon\partial_z\phi(0), \tag{5.66}$$

which is found using 5.65. If the integral of the above equation is to be zero (because it is integral of the τ derivative of a field in our setup), according to the periodicity assumption of fields, then one has $\partial_z\phi(0) = 0$. This condition in higher dimensions would give us the minimal surface condition. For the general Einstein gravity in $d+1$-dimensions, we showed in 5.41 that one can in general expand the metric of the usual $n=1$ solution around the special fixed surface as:

$$\begin{aligned} ds^2 &= dr^2 + R^2d\tau^2 + b_i d\tau dv^i + g_{ij}dv^i dv^j, \\ g_{ij} &= \gamma_{ij} + 2r\cos\tau\, K_{ij}^1 + 2r\sin\tau\, K_{ij}^2 + O(r^2) \\ R &= r + O(r^3), \quad b_i = O(r^2). \end{aligned} \tag{5.67}$$

where r is the spatial coordinate normal to the surface and v^i are coordinates along the surface. K_{ij}^α are two extrinsic curvature tensors. γ_{ij} and K_{ij}^α depend only on v^i and not on r or τ. When we deform the geometry to $n>1$ we assume that the periods of cosines above remains the same 2π. One can rewrite the metric above in the form of:

$$ds^2 = e^{2\rho}(d(x^1)^2 + d(x^2)^2) + e^{-\frac{4\phi}{d-1}}\hat g_{ij}dv^i dv^j + O(r^2), \quad det(\hat g_{ij}) = 1, \tag{5.68}$$

where $\hat{g}$ is the renormalized metric such that it's determinant is 1. If we reduce the dimensions of the above geometry to two, we arrive at the dilaton case we just discussed and we obtain the same condition $\partial_{x^\alpha}\phi = 0$ for $\alpha = 1, 2$. ϕ in the metric above is related to 5.67 by:

$$-4\phi = \ln(det(\gamma_{ij})) + x^1 K^1 + x^2 K^2 + O(r^2), \quad K^\alpha = \gamma^{ij} K^\alpha_{ij}, \tag{5.69}$$

where K^α are traces of the extrinsic curvature tensors. Now $\partial_{x^\alpha}\phi = 0$ implies:

$$\boxed{K^1 = K^2 = 0.} \tag{5.70}$$

Thus the traces of the extrinsic curvature should vanish. this is the condition which ensures that the special surface found in 5.56 is indeed minimal (a surface is minimal if and only if the trace of the second fundamental form or the extrinsic curvature is zero).

Actually one can argue for 5.70 in an easier manner. Following [27] we can define

$$z = re^{i\tau/n}, \qquad \bar{z} = re^{-i\tau/n}. \tag{5.71}$$

Then again we define the metric by working out an expansion in powers of the distance to the singular surface:

$$ds^2 = dzd\bar{z} + 2A_{iz\bar{z}}(\bar{z}dz - zd\bar{z})dv^i + (\gamma_{ij} + 2K_{ijz}z^n + 2K_{ij\bar{z}}\bar{z}^n)dv^i dv^j + \ldots, \tag{5.72}$$

where the dots denote terms that become $O(|z|^2)$ as $n \to 1$. This metric is invariant under $\mathbb{Z}_n$ transformations $z \to ze^{i2\pi s/n}$. The fixed points for the action of this symmetry form a codimension-two surface at $r = 0$ as expected. Calculating the Riemann tensor for this metric up to $O(|z|)$ gives:

$$R_{izjz} = -\frac{n-1}{z} K_{ijz} z^{n-1}, \tag{5.73}$$

which is singular at $r = 0$ for $n \sim 1$. Demanding this singularity to be harmless amounts to a finite energy-momentum tensor. Thus using

$$E_{\mu\nu}[g^{(n)}] = T_{\mu\nu} = (finite), \tag{5.74}$$

where $E_{\mu\nu}$ are gravitational field equations, One needs the Ricci scalar to be finite. Then 5.73 implies

$$\gamma^{ij} K_{ijz} = 0, \tag{5.75}$$

which is the same thing as 5.40.

So we showed that for a $\mathbb{Z}_n$ symmetric spacetime the von Neumann entropy of the gravitational density matrix is found by calculating the area of a minimal codimension-two surface which is the fixed point set of $\mathbb{Z}_n$:

$$\boxed{S_{grav} = -n\partial_n[\ln Z(n) - n \ln Z(1)]|_{n=1} = \frac{A_{minimal}}{4G_N}.} \tag{5.76}$$

This means that we proved the Area law for the gravitational entropy in quite general setup, For the Euclidean spacetimes.

5.7 Argument for the RT Formula

The proposed and proved formula 5.76 is quite general and accounts for all the Euclidean spacetimes only with the constraint of $\mathbb{Z}_n$ symmetry. Due to it's generality and the usefulness of the method of it's proof, it was beneficial to fully explain it. Now we use the same arguments to explain the RT conjecture. This argument is mainly inspired by the work of Dong [28].

As we saw before, this conjecture was in the context of AdS/CFT, and it proposes a method of finding the entropy of a region in the boundary CFT using the holography.

In order to calculate for subregion in the boundary we can proceed as we did in the case of Cardy and Calabrese's calculations and construct the appropriate replicated geometry from the boundary conformal field theory. Let us be more precise, and consider the partition function of boundary CFT with an spatial entangling region A to be $Z[M_1]$, Then by cyclically gluing together n identical copies of this system we arrive at the replicated partition function $Z[M_n]$. If CFT is holographic one may calculate, $Z[M_n]$ by finding the dominant bulk solution B_n whose asymptotic boundary is M_n. This is the statement of AdS/CFT:

$$Z[M_n] = e^{-I_{bulk}[B_n]}, \tag{5.77}$$

where $I_{bulk}[B_n]$ denotes the on-shell action of the bulk solution. Next step would be to take the Z_n orbifold of the boundary M_n ,i.e. $\tilde{M}_n = M_n/\mathbb{Z}_n$.

The orbifold boundary theory on $\tilde{M}_n$ has the same periodicity in the Euclidean time as the original M_1 so it is tempting to think that the bulk dual $\tilde{B}_n$ to this orbifold would have the action $I_{bulk}[B_1]$, but it is not completely true. In the places where we glued the original copies of M_1 to construct M_n we will have conical deficits of $2\pi(n-1)/n$ in the euclidean time in the orbifold $\tilde{M}_n$. This means that in the bulk $\tilde{B}_n$ dual to the orbifold $\tilde{M}_n$ we have to account for these conical defects. At the same time the gravitational action needs to be smooth so that the prescription of LM be applicable.
To proceed let us remember what will happen if we have a conical deficit in the Einstein gravity. The action using 5.42 is given by:

$$-\frac{1}{16\pi G_N}\left(\int_{B_1\backslash\Sigma,(0<\tau<2\pi)} R + 4\pi\frac{1}{n}(n-1)Area(\Sigma)\right) + \frac{1}{n}O((1-n)^2), \tag{5.78}$$

where Σ is the singular codimension-two singular fixed set point set of $\mathbb{Z}_n$. Thus in order to have a smooth action, we need to somehow compensate for the area term arriving due to the singularity. This can be done by introducing a comic brane with tension

$$T_n = \left(1-\frac{1}{n}\right)\frac{1}{4G_N}, \tag{5.79}$$

in the location of Σ so that we can define the action of the bulk dual to $\tilde{M}_n$ as

$$\boxed{I_{bulk}[\tilde{B}_n] = I_{bulk}[B_1] + T_n\int_\Sigma \sqrt{g},} \tag{5.80}$$

where the second term is a Nambu-Goto action for the branes. One can then readily see that the inclusion of the cosmic brane gives:

$$
\begin{aligned}
I_{bulk}[\tilde{B}_n] = &-\frac{1}{16\pi G_N}\left(\int_{B_1\backslash\Sigma,(0<\tau<2\pi)} R + 4\pi\frac{1}{n}(n-1)Area(\Sigma)\right)\\
&+ T_n\int_\Sigma \sqrt{g} + \frac{1}{n}O((1-n)^2)\\
&= -\frac{1}{16\pi G_N}\int_{B_1\backslash\Sigma,(0<\tau<2\pi)} R + \frac{1}{4nG_N}(1-n)Area(\Sigma) + \left(1-\frac{1}{n}\right)\frac{1}{4G_N}Area(\Sigma)\\
&+ \frac{1}{n}O((1-n)^2)\\
&= -\frac{1}{16\pi G_N}\int_{B_1\backslash\Sigma,(0<\tau<2\pi)} R + \frac{1}{n}O((1-n)^2),
\end{aligned} \tag{5.81}
$$

where by definition we replaced the brane actions with the area of the brane. As it can be seen the above expression does not involve the area of the singular surface anymore. Now that we have the dual action for our branched boundary, we can proceed the same way as we did before. The final action that we derived in 5.81 has periodicity of 2π in the euclidean time and the surface Σ removed. So one has to impose boundary conditions on that surface. So we arrive at the same setup that we had at 5.50, and by the same argument we have:

$$
\begin{aligned}
\partial_n I_{bulk}[\tilde{B}_n]\Big|_{n=1} &= \int_{at\ \Sigma} \frac{d^{N-2}y}{8G_N}\sqrt{g}(\nabla^\mu\partial_n g_{\mu r} - g^{\mu\nu}\nabla_r\partial_n g_{\mu\nu})\Big|_{n=1}\\
&= \frac{Area(\Sigma)}{4G_N}.
\end{aligned} \tag{5.82}
$$

The minimality condition for the surface can be found the same way that we arrived at 5.70. Thus we have for proven that the entropy is given by a quarter of the area in Planck units of a bulk codimension-two minimal cosmic brane homologous to the entangling region in the boundary.

As a remark one has to realize that the proof given above works for $\mathbb{Z}_n$ symmetric bulks in the euclidean signature. For the particular case of RT this causes no issue, because although RT's conjecture was given in the full Lorentzian signature, our proof works because for the static slices the surface found above is the same as the Lorentzian case. Furthermore this proof can be considered a generalization of the RT conjecture for the non-static spacetimes, given that we can easily move on to the Lorentzian signature.

6 Replica Wormholes, Islands and the Entropy of Hawking Radiation

In the previous chapters we reviewed various concepts involving the entanglement entropy. Particularly we talked about the information paradox and the Page's theorem, we calculated the gravitational entropy using the method proposed by LM and also we did the entanglement entropy calculation in CFTs. In 2019 it was shown that by putting these ideas together and further generalizing some of them one can propose a possible solution for the information paradox. In particular we want to review the work of Almheiri, Hartman, Maldacena, Shaghoulian and Tajdini (AHMST) [6]. In this paper AHMST demonstrates that it is possible to produce the right page curve for an evaporating black hole if one replaces the formula for entropy by a generalization of RT formula,i.e. the quantum extremal surface (QES) prescription proposed by Engelhardt and Wall [7].

As we have seen before, Hawking in his original paper [1] used the semiclassical approximation, and wrote the quantum field theory in the fixed curved black hole background and found out that black holes radiate thermally. This thermal radiation would decrease the mass of the black hole and eventually if the field theoric calculations continue to hold, the black hole will evaporate completely. Now if the initial state of the black hole and the outside radiation is pure, unitarity or the conservation of the information implies that the final state has to be pure as well. The calculations done by Hawking, imply a thermal density matrix at the end of evaporation and as a result one sees an apparent paradox which we call the Hawking's information paradox (Figure 1).

AHMST propose that in the original calculations of Hawking used the wrong formula for the entropy. They argue that in order to get the right page curve, one needs to consider saddles of higher genus in the path integral. These saddle would in turn give new contributions to the radiation subsystem which are known as islands. Consideration of islands and the use of QES prescription then would give the right page curve.

6.1 Quantum Extremal Surface Prescription

Before pressing on, we want to give a brief proof for QES prescription in a general setup. In all calculations below we consider the Euclidean signature.

We know that in order to calculate the entanglement entropy for a system, one can proceed to calculate $tr(\rho^n)$. This can be done by considering n copies of the original system. Then these copies would be glued together in a cyclic way. The resulting geometry then would be a complicated Riemann surface, but the key idea was to move on to the uniformization of this space which is basically a taking the $\mathbb{Z}_n$ orbifold of the theory. The resulting orbifold was singular at certain points and insertion of twist operators would give the the partition function. To be more precise the unnormalized correlator of twist operators can also be viewed as the partition function of the theory on a topologically non-trivial manifold, $Z_n = \langle \mathcal{T}_1 ... \mathcal{T}_k \rangle$. Then by the replica method with the analytic continuation in n on has:

$$S = -\partial_n \left(\frac{\ln Z_n}{n} \right) \Bigg|_{n=1}. \tag{6.1}$$

Let us consider a manifold $\tilde{\mathcal{M}}_n$ which is constructed by cyclically gluing of n identical copies of an original setup of fields and geometry $\mathcal{M}_1$. This manifold may consist of regions with dynamical gravity and non-gravitational regions, on which geometry is fixed. For example later we shall consider an eternal AdS_2 black hole glued to a fixed Minkowski spacetime in both sides. The matter fields in the non-gravitational part are coupled to a fixed background geometry and in the gravitational parts any manifold, with any topology, which obeys the appropriate boundary conditions must be considered in the gravitational path integral. This consideration of different topologies is the reason for appearance of the replica wormholes in addition to the usual AdS_2 gravity in our calculations later on. With all these considerations one can see that the full action of the theory is the sum of the gravitational action and the partition function of the quantum fields on the geometry $\tilde{\mathcal{M}}_n$ (divided by n for ease in the calculations):

$$\boxed{\frac{\ln Z_n}{n} = -\frac{1}{n} I_{grav}[\tilde{\mathcal{M}}_n] + \frac{1}{n} \ln Z_{mat}[\tilde{\mathcal{M}}_n].} \tag{6.2}$$

This action is smooth with the Euclidean time τ going from 0 to $2\pi n$. One must view this action as an effective action for the geometry, meaning the integral over geometries is evaluated at saddle points. The metric is classical, but one includes quantum corrections by inserting the quantum expectations value of the matter stress tensor on the geometry.

Another essential assumption is the replica symmetry. This means that metric is invariant on $\tilde{\mathcal{M}}_n$ under the action of the group $\mathbb{Z}_n$ on τ coordinate: $\tau \sim \tau + 2\pi k,\ \ k \in \{1, 2, ..., n\}$.
The replica symmetry allows one to consider the equivalence relation between points for which τ coordinates differ by $2\pi k$. Then it is possible to take the $\mathbb{Z}_n$ orbifold and construct the quotient topology $\mathcal{M}_n = \tilde{\mathcal{M}}_n/Z_n$ on which τ runs from 0 to 2π. This quotient manifold is a manifold with conical deficit singularity at fixed points of the $\mathbb{Z}_n$ action on the original $\tilde{\mathcal{M}}_n$. Roughly speaking one can imagine that $\tilde{\mathcal{M}}_n$ has a metric at small r coordinates in the form of

$$ds^2 = n^2 dr^2 + r^2 d\tau^2 \dots . \tag{6.3}$$

by introducing $R = r/n$ and $\phi = \tau/n$ one has:

$$ds^2 = dR^2 + R^2 d\phi^2, \tag{6.4}$$

which is regular at all points including $R = 0$ due to the fact that τ runs from 0 to $2\pi n$ (ϕ runs from 0 to 2π). Now if we consider the quotient space $\mathcal{M}_n$ then we have a conical deficit of $2\pi(n-1)/n$ or equivalently an opening angle of $2\pi/n$. These points of singularity will correspond to the endpoints of the island region. We may also have regions in the non-gravitational part of the manifold, and those will involve endpoints as well. In order to perform a calculation similar to that of Cardy and Calabrese, one needs to insert twist operators in both fixed points of $\mathbb{Z}_n$ in the gravitational part and also endpoints of entanglement regions in the non-gravitational parts. This will give us the matter contribution to the entropy.

To find the gravitational contribution to the entropy one needs to know how to relate the gravitational action of $\mathcal{M}_n$ to that of $\tilde{\mathcal{M}}_n$. It might be tempting to consider $I_{grav}[\tilde{\mathcal{M}}_n] = nI_{grav}[\mathcal{M}_n]$, but this is not right since the action $I_{grav}[\tilde{\mathcal{M}}_n]$ is smooth while the action $I_{grav}[\mathcal{M}_n]$ involves conical singularities. It is now apparent that our situation is similar to that of 5.80.

Indeed again we use the idea of Dong [28] and introduce cosmic branes into the quotient manifold $\mathcal{M}_n$ to compensate for the conical singularities. These cosmic branes are codimension-two surfaces with suitable tensions

$$T_n = \left(1 - \frac{1}{n}\right)\frac{1}{4G_N}, \tag{6.5}$$

and are located in points of conical singularity or the fixed points of $\mathbb{Z}_n$. Then the full gravitational part of the action would be:

$$\boxed{\frac{I_{grav}}{n}[\tilde{\mathcal{M}_n}] = I_{grav}[\mathcal{M}_n] + T_n \int_\Sigma \sqrt{g},} \tag{6.6}$$

where the second term is a Nambu-Goto action for the branes we considered, and Σ is the singular set.

Since we have conical singularities in $\mathcal{M}_n$, we can use 5.42 and write:

$$\begin{aligned} I_{grav}[\mathcal{M}_n] = &-\frac{1}{16\pi G_N}\left(\int_{\tilde{\mathcal{M}_n}\backslash\Sigma,(0<\tau<2\pi)} R + 4\pi\frac{1}{n}(n-1)Area(\Sigma)\right) \\ &+ \frac{1}{n}O((1-n)^2) + I_{boundary}[\partial\mathcal{M}_n], \end{aligned} \tag{6.7}$$

where Σ is the singular codimension-two surface or the cosmic brane and integral on Ricci scalars is done from $\tau = 0$ to $\tau = 2\pi$ in $\tilde{\mathcal{M}_n}$. The term $I_{boundary}$ consists of all possible boundary actions, such as the extrinsic curvature terms that one encounters in JT gravity or the Gibbons-Hawking term. As mentioned in the discussion of the squashed cones, the $O((1-n)^2)$ terms are relevant only for more than two-dimensional spacetimes.So integrating the action 6.6 and using 6.7 one has:

$$\begin{aligned} \frac{I_{grav}}{n}[\tilde{\mathcal{M}_n}] = &-\frac{1}{16\pi G_N}\left(\int_{\tilde{\mathcal{M}_n}\backslash\Sigma,(0<\tau<2\pi)} R + 4\pi\frac{1}{n}(n-1)Area(\Sigma)\right) \\ &+ T_n\int_{\Sigma_{D-2}}\sqrt{g} + \frac{1}{n}O((1-n)^2) + I_{boundary}[\partial\mathcal{M}_n] \\ &= -\frac{1}{16\pi G_N}\int_{\tilde{\mathcal{M}_n}\backslash\Sigma,(0<\tau<2\pi)} R + \frac{1}{4nG_N}(1-n)Area(\Sigma) \\ &+ \left(1-\frac{1}{n}\right)\frac{1}{4G_N}Area(\Sigma) + \frac{1}{n}O((1-n)^2) + I_{boundary}[\partial\mathcal{M}_n] \\ &= -\frac{1}{16\pi G_N}\int_{\tilde{\mathcal{M}_n}\backslash\Sigma,(0<\tau<2\pi)} R + \frac{1}{n}O((1-n)^2) + I_{boundary}[\partial\mathcal{M}_n], \end{aligned} \tag{6.8}$$

which shows that indeed the contribution of the comic brane and conical singularities cancel out and the gravitational action is smooth as it should be. However one has to set boundary conditions as one approaches to the points of Σ because these points got omitted from the integration of Ricci scalar and we will have non-vanishing boundary variation. Then like the calculations of 5.51 consideration of this boundary leads to:

$$\begin{aligned} \partial_n\left(\frac{I_{grav}}{n}[\tilde{\mathcal{M}_n}]\right)\Bigg|_{n=1} &= \int_{at\ \Sigma}\frac{d^{N-2}y}{8G_N}\sqrt{g}(\nabla^\mu\partial_n g_{\mu r} - g^{\mu\nu}\nabla_r\partial_n g_{\mu\nu})\Bigg|_{n=1} \\ &= \frac{Area(\Sigma)}{4G_N}, \end{aligned} \tag{6.9}$$

where y^i are the coordinates along the cosmic brane and N is the spacetime dimension.

Let us rewrite the action 6.2 with the above considerations:

$$\boxed{\frac{I^{tot}}{n} = \frac{\ln Z_n}{n} = -\Big(I_{grav}[\mathcal{M}_n] + T_n \int_{\Sigma} \sqrt{g}\Big) + \frac{1}{n} \ln Z_{mat}[\tilde{\mathcal{M}}_n].} \tag{6.10}$$

When $n = 1$ we have the manifold $\mathcal{M}_1 = \tilde{\mathcal{M}}_1$, which is the original solution to the problem. It is a solution of the I^{tot}. Next we taylor expand the action around the $n = 1$:

$$\left(\frac{I^{tot}}{n}\right)_{n\to 1} = I_1^{tot} + (n-1)\partial_n \left(\frac{I^{tot}}{n}\right)\Bigg|_{n=1}. \tag{6.11}$$

Now it is instructive to look at $\partial_n \left(\frac{I^{tot}}{n}\right)\Big|_{n=1}$ more carefully, keeping in mind the relation 6.1. This term involves two contributions. first contribution comes from the parenthesis in the right side of the 6.10, which in turn gives the area of the cosmic brane as we calculated in 6.9. The second contribution comes from the matter term in 6.10, which gives the entropy due to matter fields. For example in the case of $2D$ CFTs this term involves calculations reminiscent to that of Cardy and Calabrese. So we have

$$\boxed{-\partial_n \left(\frac{I^{tot}}{n}\right)\Bigg|_{n=1} = \frac{Area}{4G_N} + S_{matter}.} \tag{6.12}$$

This expression is called the generalized entropy and is denoted by $S_{gen}(w_i)$, where we denoted the positions of the cosmic branes by w_i to emphasize that this entropy depends on them.

Now we must remember that we started from $\mathcal{M}_1 = \tilde{\mathcal{M}}_1$ which is a solution for the original action I_1^{tot}. So if we calculate the variation of 6.11 at $g^{(1)}$ which is the metric for $\tilde{\mathcal{M}}_1$ we have:

$$\begin{aligned} \delta\left(\frac{I^{tot}}{n}\right)_{n\to 1} &= \delta I_1^{tot} + (n-1)\delta\left(\partial_n \left(\frac{I^{tot}}{n}\right)\Bigg|_{n=1}\right) \\ &= (n-1)\delta\left(\partial_n \left(\frac{I^{tot}}{n}\right)\Bigg|_{n=1}\right) = (1-n)\delta S_{gen}(w_i) = 0. \end{aligned} \tag{6.13}$$

This means that the real entropy is the one which is extremizes the S_{gen}. Now we derived the quantum extremal surface (QES) condition for calculation of the entanglement entropy:

$$\boxed{S = Ext\, S_{gen} = Ext\left(\frac{Area(\Sigma)}{4G_N} + S_{matter}\right).} \tag{6.14}$$

Let us recap what we did. First we considered a geometry $\tilde{\mathcal{M}}_1$ which is the solution of the action 6.2 for $n = 1$. We replicated the geometry to n copies and constructed the orbifold geometry. then introduced a cosmic brane to analytically continue the replication to non-integer n and and omit the singular points from the integral of the action. Finally we arrived at 6.10. Then in order to find the entanglement entropy we expanded the action to first order in $n-1$ and then extremized the terms of order $n-1$. The entropy is given by this extremized value.

As a last remark we have to emphasize that in the arguments above we considered the Einstein gravity because we assumed that the action is proportional to Ricci scalar. For the cases of higher derivative gravities one may need to add extra terms to cosmic brane action in order to get the desirable result.

6.2 Geometry of Two Dimensional Eternal Black Holes and the Information Paradox

Now it is time to introduce the setup of the calculations that we shall use. A simple setup to consider is the AdS_2 eternal black hole geometry, But there is a caveat. This kind of black hole does not radiate. In fact at the asymptotic boundary there exists reflective surfaces that reflect light rays back into the geometry [29]. In order to study the information problem, one has to devise a setup that allows for the radiation to happen. In this case one can glue the black hole from two sides to the flat space where the gluing happens in a distance less than the reflective boundary of the AdS_3 and a transparent wall with Dirichlet boundary conditions imposed on it is placed at the gluing boundary. The Penrose diagram for this setup in given in figure 14. Note that in all of the figures of this chapter, the shaded regions are the regions with dynamical gravity, and the white regions are endowed with constant Minkowski space. For the radiation to happen we consider that in all of the geometry there exists a 2-dimensional CFT. We will give the specifics of the action of gravity and matter as well as the gluing process later on.

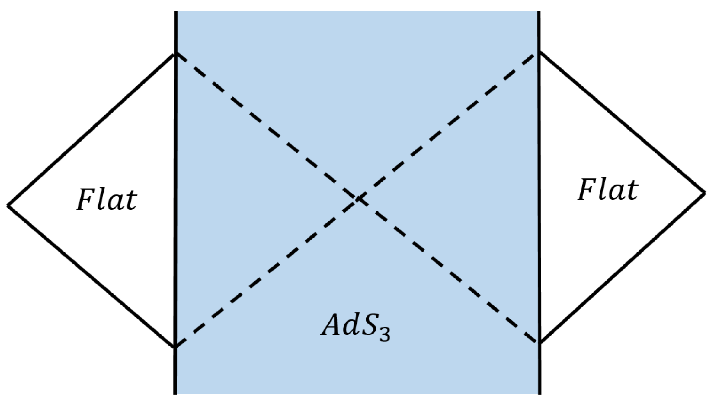


Figure 14: The evaporating eternal two-dimensional black hole

In the Euclidean signature, This geometry can viewed as a cylinder. This is because as we have seen before, AdS_2 in the Euclidean signature is equivalent to a Poincare disk and finite temperature flat space geometry is a cylinder. This cylinder can be transformed to the complex plane with a disk removed from it. In fact we have seen this transformation in the dicussion of entanglement entropy in conformal field theories at finite temperature in the previous chapters. On the Euclidean geometry we can define the state of the system to be the thermofield double:

$$|state\rangle = |TFD\rangle = |0(t=0)\rangle \propto \lim_{s\to\infty} e^{-Hs}\,|\phi(t=-s)\rangle \propto \int_{\phi(t\to-\infty)}^{\phi(t=0)=\square} [d\phi]\, e^{-I_E}, \tag{6.15}$$

Where H is the hamiltonian of the system and $\square$ in the upper limit of the path integral indicate that we have a functional which needs field input in the upper limit, i.e. this state is a wave functional. Schematically we can view this as integrating the lower half of the geometry (see figure 15).

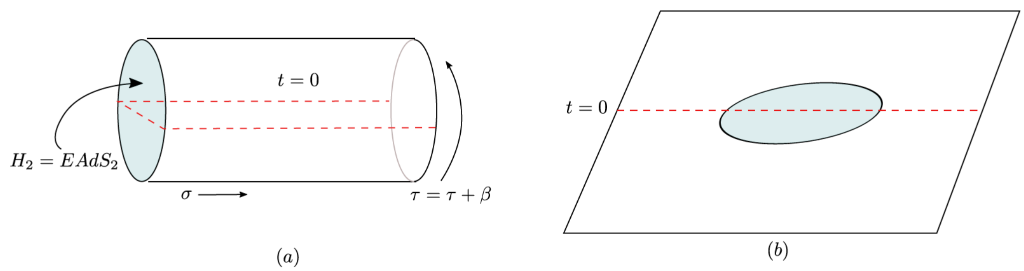


Figure 15: The thermofield double state preparation in (a) cylindrical demonstration and (b) the equivalent plane demonstration. These figures are taken from [6].

For this geometry one still has the information paradox and the Hawking calculation gives increasing entropy as with the cases that we discussed previously.

6.3 The Replica Wormholes and Islands

We mentioned briefly that the basis of AHMST's argument lies in considering new saddles. But where do these saddles come from? and how do they contribute entropy of the radiation? In order to answer these questions we have to construct the replica geometry by cyclic identification of the radiation subspaces. This is means that in the flat region of the geometry we have to radiation subspaces that are identified, the same setup that one had in the calculations of Cardy and Calabrese. But there is a catch. Unlike the usual conformal field theory setup, here we allowed the geometry inside the Poincare disk to be the dynamical AdS_2 gravity. This means that any solution that satisfies the appropriate boundary condition is accepted. In particular we have the two saddles in figure 16. The first saddle is the usual Hawking saddle for which the gravitational region is separated. The other saddle is the one which is called the "replica wormhole".

We are considering the Einstein gravity in which the action is of the form $I = S_0 \int R$. In two dimensions the Gauss-Bonnet theorem of the differential geometry states that $\int R = \chi = 2 - 2g$, where χ is the Euler characteristic of the geometry and g is the genus. So if we expand the gravitational path integral in the semiclassical approximation, each saddle will be suppressed by

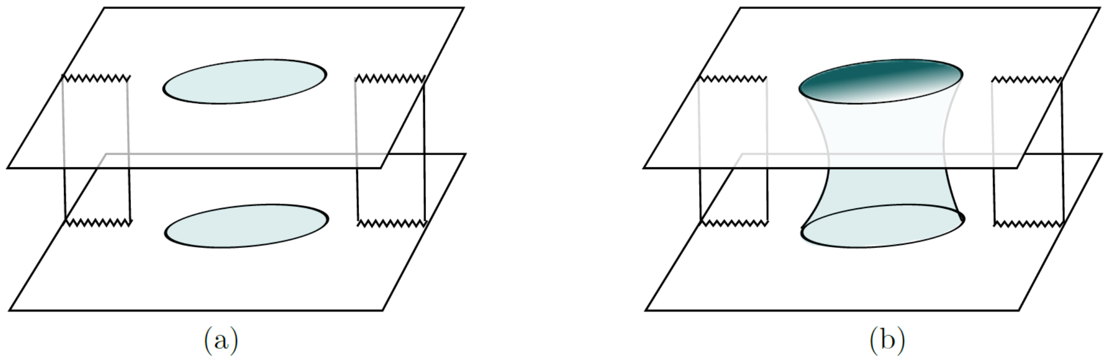


Figure 16: (a) The Hawking and (b) replica wormhole saddles, which are both acceptable solutions. These figures are taken from [6].

it's genus:

$$\boxed{\ln Z \approx \sum_{\gamma} \chi(\gamma) S_0,} \tag{6.16}$$

where γ are different saddles. In our calculation we consider only the two saddles mentioned above. There is no guarantee that these two are the only ones that are contributing, but the thing that seems to be right is that they give the right page curve for the entropy of radiation.

The replica wormholes that we consider are the ones that are replica symmetric. This means that they have a fixed point around which one can rotate the coordinates of the geometry so that nothing changes. This is the same $\mathbb{Z}_n$ symmetry that we have seen a lot before. These wormholes have genus $g = n - 1$ which can be found using the Riemann-Hurwitz formula. So their effect is suppressed by their genus and usually one neglects these kinds of saddles. However as we shall see in the late times these saddles become dominant and this can be seen from their contribution to the entropy according to the QES prescription.

Now we address to the question in regard to contribution of wormholes saddles to the entropy. To get a clearer picture, we use the replica symmetry and take the $\mathbb{Z}_n$ orbifold of the wormhole geometries. The resulting orbifold action will have singularities at fixed points of $\mathbb{Z}_n$, on which one needs to introduce cosmic branes. The orbifold is found by limiting the angular coordinate of the geometry to $2\pi/n$ (see figure 18).

In particular the behavior of the orbifold at $n \sim 1$ is of our interest. for that we need to widen the $2\pi/n$ opening angle so that it is close to 2π it is rather hard to picture but after some imagination one can see that we will arrive at figure 19.

This is interesting because it points out that in the cases where we have replica wormholes, in the orbifold we need another identification between between the fixed points of the $\mathbb{Z}_n$ or the conical singularities in addition to the usual identification of the radiation subspace. This additional region that needs to be considered is called an "entanglement island".

To summarize we considered two saddles that contribute to the gravitational partition function.

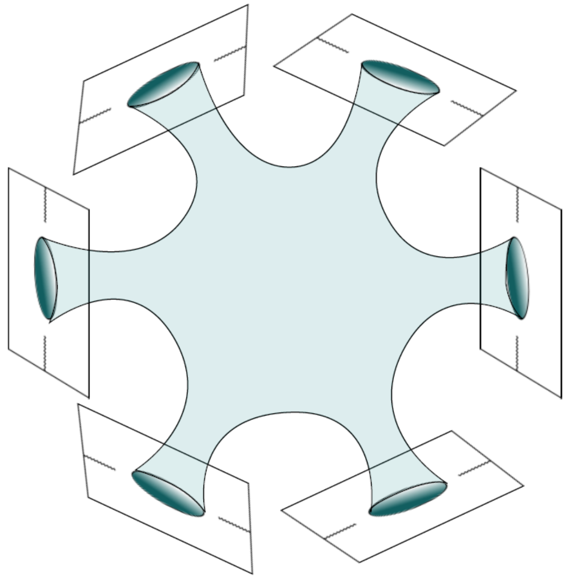


Figure 17: A $\mathbb{Z}_6$ symmetric replica wormhole geometry. This figure is taken from [6].

The first saddle is the usual Hawking saddle which is the black hole. And the second one is the replica wormhole geometry which gives island contributions to the entropy. So after taking the $\mathbb{Z}_n$ orbifold and the limit $n \to 1$, we will have two setups for which we have to calculate the entropy. These are shown in figure 20 in which the w_1, w_2 are fixed points of $\mathbb{Z}_n$, I denotes the island region and R are radiation regions. the setup (a) is the one found from the wormhole saddles and (b) is the usual Hawking saddle.

6.4 The Two Dimensional JT Gravity Theory Plus a CFT

Now that we have the general picture of what is going on, we can work out the calculations a bit more precise. For the dynamical geometry part, we consider the JT gravity model that is glued to the Minkowski space in the outside to allow for radiation. We also have a CFT which lives in the whole JT+Minkoswki geometry. We have the same CFT in the interior JT and exterior flat geometry, so we can impose transparent boundary conditions, see figure 21. The suitable action for this setup is the following action

$$\ln Z^{tot} = \frac{S_0}{4\pi}\left[\int R + \int_{boundaries} 2K\right] + \int \frac{\phi}{4\pi}(R+2) + \frac{\phi_b}{4\pi}\int_{boundaries} 2K + \ln Z_{CFT}[g], \tag{6.17}$$

where we considered a JT model with boundary in the gravitational part of the action in order to impose the transparent boundary conditions we mentioned. In the above action we set $4G_N = 1$ so that the area terms in the entropies will be just given by the value of ϕ, $\frac{Area}{4G_N} = S_0 + \phi$, due to the fact that the codimension-two cosmic branes are just points in this example.

Then we introduce the replica geometry $\tilde{\mathcal{M}}_n$ on which τ goes from 0 to $2\pi n$. Then we go to the quotient space $\mathcal{M}_n = \tilde{\mathcal{M}}_n/\mathbb{Z}_n$ which is possible due to the assumption of $\mathbb{Z}_n$ symmetry.

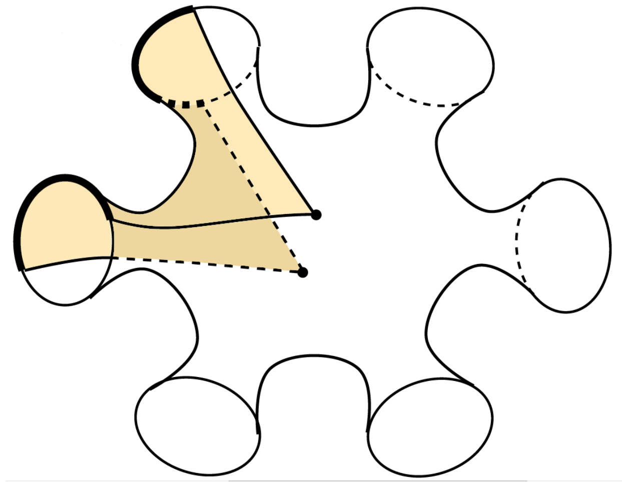


Figure 18: The $\mathbb{Z}_6$ orbifold of the replica wormhole saddle is the shaded region. This figure is taken from [8].

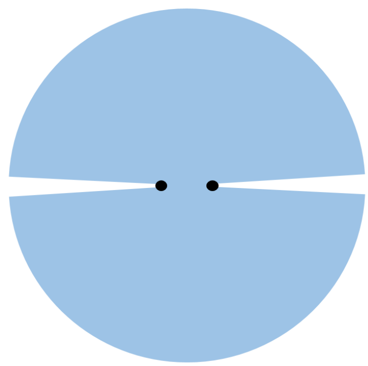


Figure 19: The $\mathbb{Z}_n$ orbifold of the replica wormholes at $n \sim 1$.

Translating the calculations in $\mathcal{M}_n$ simplifies the description of the manifold as we saw in the last section. This manifold can be viewed as a hyperbolic disk with conical singularities and with twist operators for the matter theory inserted at these singularities. These conical singularities can be realized with consideration of cosmic branes in the original geometry following 6.6, which would result a gravitational action of the form (remember we set $4G_N = 1$)

$$\boxed{-\frac{I_{grav}}{n} = \frac{S_0}{4\pi}\left[\int_{\mathcal{M}_n} R + \int_{\partial\mathcal{M}_n} 2K\right] + \int_{\mathcal{M}_n} \frac{\phi}{4\pi}(R+2) + \frac{\phi_b}{4\pi}\int_{\partial\mathcal{M}_n} 2K - (1-\frac{1}{n})\sum_i [S_0 + \phi(w_i)],} \tag{6.18}$$

where w_i are the positions of the cosmic branes which are just instantons or -1 branes in this case. We also know that for the action above τ integrals are done from 0 to 2π on $\mathcal{M}_n$ geometry. This geometry is the one on which we introduce n identical copies of the CFT, to repeat the calculations of Cardy and Calabrese. On the cosmic branes we put twist fields. The positions of these cosmic branes are dynamical. We evaluate the action of the CFT as a quantum theory and then insert the expectation value of it's stress energy tensor in the gravitational equations of motion. This

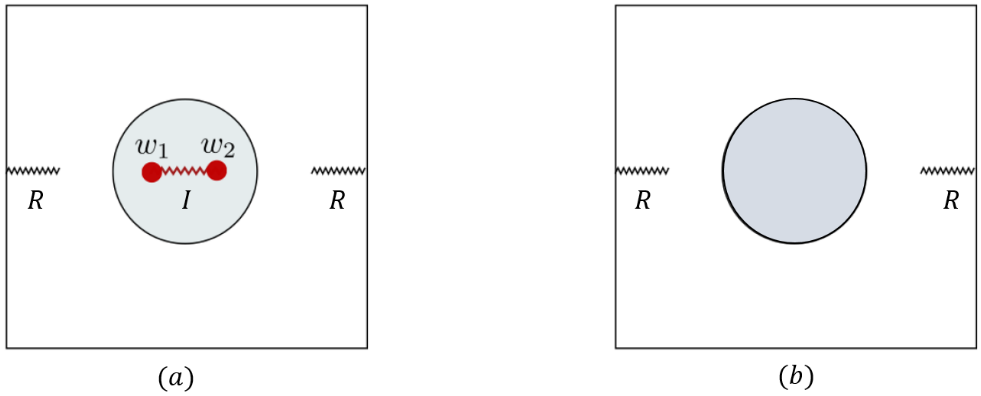


Figure 20: (a) The region $R \cup I$ contributes to the entropy when there are wormholes. (b) for the usual Hawking saddle only R contributes to the entropy.

approximation is justified if we assume $c \gg 1$ so that the graviton fluctuations are suppressed.

We can define an interior complex coordinate w where the metric for the manifold $\mathcal{M}_n$ in the gravitational region is

$$ds^2 = e^{2\rho} dw d\bar{w}, \quad with \ |w| \leq 1. \tag{6.19}$$

This is a general form of a hyperbolic disk with singularity. The information in regard to the singularities is encapsulated in the ρ parameter. The boundary of AdS_2 is at $|w| = 1$, or $w = e^{i\theta}$ for some $\theta \in \mathbb{R}$. This choice of metric is further justified by looking at the dilaton equation of motion and looking at how the final action would look like if we impose this equation. The dilaton equation of motion is obtained by varying 6.18:

$$\frac{\sqrt{g}}{4\pi}(R+2) - (1-\frac{1}{n})\sum_i \delta^2(w-w_i) = 0, \tag{6.20}$$

where the $|g|$ is included by considering the right integral measure beside the integrands in the action. If we plug in the metric 6.19 into this equation, we obtain the following using the Weyl transformation of Ricci scalar:

$$\begin{aligned} &\frac{1}{4\pi}e^{2\rho}(e^{-2\rho}(-2\nabla^2\rho)+2) = (1-\frac{1}{n})\sum_i \delta^2(w-w_i) \\ &\implies -\nabla^2\rho + e^{2\rho} = 2\pi(1-\frac{1}{n})\sum_i \delta^2(w-w_i) \\ &\implies -4\partial_\omega\partial_{\bar{\omega}}\rho + e^{2\rho} = 2\pi(1-\frac{1}{n})\sum_i \delta^2(w-w_i). \end{aligned} \tag{6.21}$$

where in the last line we used the definition of complex partial derivatives, $\partial_w = \frac{1}{2}(\partial_x - i\partial_y)$.

Now if we impose this equation in the action 6.18 we will see that the delta functions of curvature

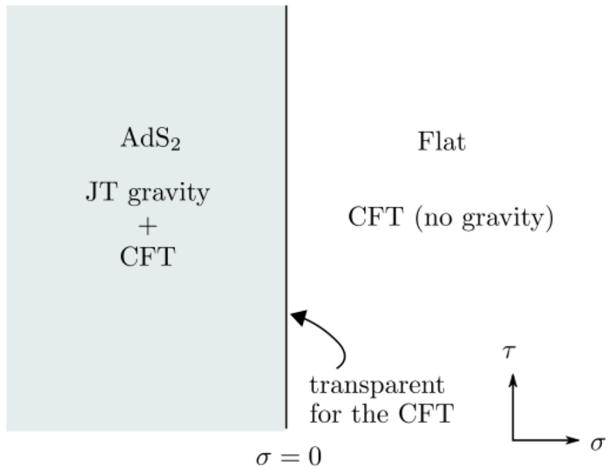


Figure 21: The setup we consider which consists of JT gravity for $\sigma < 0$ and the Minkowski geomtery for $\sigma > 0$. The same CFT lives in both sides. This figure is taken from [6].

cancel against the cosmic brane contributions:

$$
\begin{aligned}
\frac{S_0}{4\pi}\int_{\mathcal{M}_n} R + \int_{\mathcal{M}_n} \frac{\phi}{4\pi}(R+2) - (1-\frac{1}{n})\sum_i [S_0 + \phi(w_i)] = \\
S_0 \int dw d\bar{w}\, \frac{1}{2\pi}(-4\partial_w \partial_{\bar{w}}\rho) + \int dw d\bar{w}\, \frac{\phi}{2\pi}(-4\partial_w\partial_{\bar{w}}\rho + e^{2\rho}) \\
-(1-\frac{1}{n})\sum_i [S_0 + \phi(w_i)] = \\
S_0 \int dw d\bar{w} \left((1-\frac{1}{n})\sum_i \delta^2(w-w_i) - \frac{e^{2\rho}}{2\pi}\right) + \int dw d\bar{w}\, \phi(1-\frac{1}{n})\sum_i \delta^2(w-w_i) \\
-(1-\frac{1}{n})\sum_i [S_0 + \phi(w_i)] = \\
-S_0 \int dw d\bar{w}\, \frac{e^{2\rho}}{2\pi} + (1-\frac{1}{n})\sum_i [S_0 + \phi(w_i)] - (1-\frac{1}{n})\sum_i [S_0 + \phi(w_i)] = \\
-S_0 \int dw d\bar{w}\, \frac{e^{2\rho}}{2\pi} = \frac{S_0}{4\pi}\int_{\mathcal{M}_n \backslash (singularities)} R.
\end{aligned}
\tag{6.22}
$$

In the equations above we used the fact that from 6.21 and Weyl transformation of Ricci scalar one has for our metric:

$$
R = -2e^{-2\rho}(\nabla^2 \rho) = 4\pi e^{-2\rho}(1-\frac{1}{n})\sum_i \delta^2(w-w_i) - 2. \tag{6.23}
$$

which in particular means that the last in equality in 6.22 is the integral of Ricci scalar without considering the curvature delta functions. This is important, because it is an explicit example of the procedure in which one can regularize the action by introduction of cosmic branes and arriving at 6.19. This also shows that our choice of metric is rather suitable for this problem. The final form of the gravitational action by considering 6.22 is:

$$\boxed{-\frac{I_{grav}}{n} = \frac{S_0}{4\pi}\left[\int_{\mathcal{M}_n \backslash (singularities)} R + \int_{\partial \mathcal{M}_n} 2K\right] + \frac{\phi_b}{4\pi}\int_{\partial \mathcal{M}_n} 2K,} \tag{6.24}$$

which is the same as 6.8 and is the suitable expression we expected to find.

Next we add the non-gravitational part of the geometry. It is a special case of the mathematical problem of conformal welding which is connecting two Riemann surfaces to construct another Riemann surface. we define coordinates σ, τ on the Minkowski region in the outside so that at the boundary we have $\sigma = 0$. We then go to complex coordinates $y = \sigma + i\tau, \bar{y} = \sigma - i\tau$ and define $v = e^y$. We consider the finite temperature case so we have $\tau \sim \tau + 2\pi$. Then the physical half cylinder $\sigma \geq 0$ corresponds to $|v| \geq 1$. At the boundary we have $v = e^{i\tau}$. This boundary must connect to the boundary of the gravitational part of geometry on which we have $w = e^{i\theta}$. So if we want to connect these two, we have $v = e^{i\tau}, w = e^{i\theta(\tau)}$ at the boundary. There is no need for our original coordinates v, w to continue holomophically from one region to the other. More precisely we need to find two functions F and G such that they satisfy

$$\begin{aligned} z &= G(w), \quad |w| \leq 1 \\ z &= F(v), \quad |v| \geq 1 \\ G(e^{i\theta(\tau)}) &= F(e^{i\tau}), \quad |w| = |v| = 1, \end{aligned} \tag{6.25}$$

and be holomorphic in their respective domains (they do not have to be holomorphic at the boundary). The problem of finding these functions is called the conformal welding problem. If we find such functions then the coordinate z in the complex plane covers the whole Riemann surface meaning that two regions in the complex plane correspond to the our first two Riemann surfaces. The functions F and G end up depending non-locally on $\theta(\tau)$ (we will see it explicitly in an example that we will calculate) and they map the inside and outside disks to the inside and outside of some irregular region in the complex plane. To visualize the things we said see figure 22.

Let us talk about the exact behavior of these functions a bit more. The welding maps go to the complex plane with the z coordinates. On this plane one can imagine doing $SL(2, \mathbb{C})$ transformations and the welding maps F, G are ambiguous up to these transformations. These transformations move around the point at infinity in $\mathbb{C}$, and thus we would need to allow a pole at infinity for the functions F and G. But one knows that a function with a pole at infinity can be extended to a holomorphic function on the Riemann sphere. From the stereographic projection, it can be seen that the Riemann sphere is the complex plane plus a point at infinity, i.e. $\mathbb{C} \cup \{\infty\}$. So we fix $F(\infty) = \infty$ in our welding maps and by doing so the resulting function would be holomorphic everywhere on the Riemann sphere, with no poles. Then the group of transformations is reduced to just translations, scalings and rotations of the plane z, meaning that by fixing the value of F at the point of infinity we no longer have the ambiguity of transformations of this point and this point is now fixed in the image of the welding maps.

Now the aim is to solve for F, G perturbatively. This means that we consider that the gluing function is given in the form $\theta(\tau) = \tau + \delta\theta(\tau)$, where $\delta\theta(\tau)$ is a fixed input to the problem which

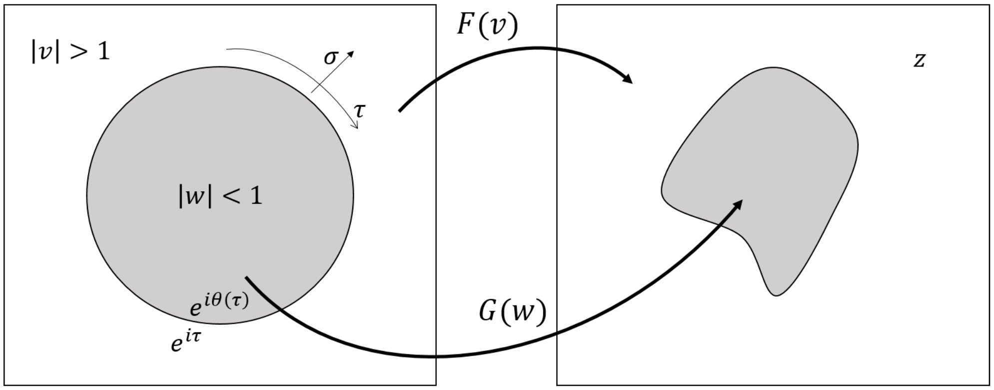


Figure 22: The conformal welding. We are given two Riemann surfaces with two coordinates $|w| \leq 1$ and $|v| \geq 1$ with boundaries glued in terms of a function $\theta(\tau)$ where $e^{i\theta}$ and $v = e^{i\theta(\tau)}$. Then we need to find holomorphic maps of each of these regions to regions in complex plane with coordinate z so that they are compatible at the boundary.

is small so that $\theta(\tau)$ is close to the identity. Since as explained above F, G are holomorphic on the Riemann sphere, we have Laurent expansions

$$G(w) = w + \sum_{l=0}^{\infty} g_l w^l, \qquad F(v) = v + \sum_{l=-\infty}^{2} f_l v^l. \tag{6.26}$$

And the function $\theta(\tau)$ is periodic so it has the following Fourier expansion:

$$\theta(\tau) = \tau + \sum_{m=-\infty}^{\infty} c_m e^{im\tau}. \tag{6.27}$$

In all of the expansions above the coefficients are considered to be small. As we said above there are still $SL(2,\mathbb{C})$ transformations that translate the points in the z plane, and by using them we gauge-fixed the zeroth order of these maps to the identity. The data of the welding problem is encapsulated in the matching condition

$$G(e^{i\theta(\tau)}) = F(e^{i\tau}) \implies e^{i\theta(\tau)} + \sum_{l=0}^{\infty} g_l e^{il\theta(\tau)} = e^{i\tau} + \sum_{l=-\infty}^{2} f_l e^{il\tau}. \tag{6.28}$$

We can expand this equation, to find:

$$(1 + i\delta\theta(\tau))e^{i\tau} + \sum_{l=0}^{\infty} g_l \Big((1 + i\delta\theta(\tau))e^{i\tau} \Big)^l = e^{i\tau} + \sum_{l=-\infty}^{2} f_l e^{il\tau} \tag{6.29}$$

Then in the linearized level (first order in coefficients c_m, g_l and f_l) for the modes with $l > 2$:

$$f_l e^{il\tau} = 0 = g_l e^{il\tau} + ic_{l-1} e^{i(l-1)\tau} \times e^{i\tau} \implies g_l = -ic_{l-1}. \tag{6.30}$$

And for the modes with $l < -2$:

$$ic_{l-1} e^{i(l-1)\tau} \times e^{i\tau} + 0 = f_l e^{il\tau} \implies f_l = ic_{l-1}. \tag{6.31}$$

And for $l = 2$:

$$g_2 e^{2i\tau} + ic_1 e^{i\tau} \times e^{i\tau} = f_2 e^{i\tau} \implies ic_1 = f_2 - g_2. \tag{6.32}$$

And for $l = 1$:

$$(1 + ic_0)e^{i\tau} + g_1 e^{i\tau} = e^{i\tau} + f_1 e^{i\tau} \implies ic_0 = f_1 - g_1. \tag{6.33}$$

And finally for the modes with $l = 0$:

$$ic_{-1} e^{-i\tau} \times e^{i\tau} + g_0 = f_0 \implies ic_{-1} = f_0 - g_0. \tag{6.34}$$

So in total we found that:

$$\begin{gathered} f_{l+1} = ic_l \quad (l \leq -2) \\ g_{l+1} = -ic_l \quad (l \geq 2) \\ ic_{-1} = f_0 - g_0, \;\; ic_0 = f_1 - g_1, \;\; ic_1 = f_2 - g_2. \end{gathered} \tag{6.35}$$

Then we can further use the $SL(2, \mathbb{C})$ ambiguity to fix $G(0) = 0, \; F(v) = v + \mathit{constant}$ as $v \to \infty$. This amounts to setting

$$g_0 = f_1 = f_2 = 0 \tag{6.36}$$

in 6.26 and 6.27. Putting these all together we have unique solutions for F and G in terms of $\delta\theta(\tau)$, by finding f_l, g_l in terms of the c_m

$$\boxed{f_l = ic_{l-1}, \quad l \leq 0, \qquad g_l = -ic_{l-1}, \quad l > 0.} \tag{6.37}$$

Thus we can calculate at the linearized level:

$$\begin{aligned} \{F, v\} &= \left(\frac{F'''}{F'}\right) - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 \\ &= \left(\frac{\sum_{l=-\infty}^{-2}(l+1)l(l-1)ic_l v^{l-2}}{1 + ...}\right) - \frac{3}{2}\frac{O(c^2)}{1 + ...} \\ &= \sum_{l=-\infty}^{-2} l(l^2 - 1)ic_l v^{l-2}. \end{aligned} \tag{6.38}$$

Note that in the above equation, derivatives are with respect to v. Now by definition of the first variation one has:

$$\delta\{e^{i\theta(\tau)}, \tau\} := \{e^{i\tau + i\delta\theta}, \tau\} - \{e^{i\tau}, \tau\} = \delta\theta''' + \delta\theta', \tag{6.39}$$

where now the derivatives are with respect to τ. From 6.27 one has that $\delta\theta(\tau) = \sum_{m=-\infty}^{\infty} c_m e^{im\tau}$ so we have

$$\delta\{e^{i\theta}, \tau\} = -i \sum_{m=-\infty}^{\infty} m^3 c_m e^{im\tau} + i \sum_{m=-\infty}^{\infty} m c_m e^{im\tau} = -\sum_{m=-\infty}^{\infty} m(m^2 - 1)ic_m e^{im\tau}. \tag{6.40}$$

Comparing this equation with 6.38 one sees that at the boundary of the interior and the exterior:

$$e^{2i\tau}\{F, v\} = -\delta\{e^{i\theta}, \tau\}_- = -(\delta\theta''' + \delta\theta')_-, \tag{6.41}$$

where the negative subscript means that the functions are projected onto negative-frequency Fourier modes. We will comeback to this result later on.

6.5 Single Interval at Finite Temperature

Now we tend to the actual computation of the entropy. First we consider a single interval. This will give us hints on how to generalize the result to two intervals. We consider a single interval $[0, b]$ that ends in the boundary between the gravitational and flat part of the geometry. Turning on the dynamics in the gravitational part will change the point 0 to a point $-a$ in the gravitational part very much like the way islands appeared in the discussion of the previous sections. Thus we need to consider an interval $[-a, b]$ in $\mathcal{M}_n$. First we do the calculations using the QES method we introduced earlier. Then we derive the results directly by replica method and replica wormholes in gravity.

6.5.1 QES calculation for entropy

Let us consider an eternal black hole, glued to flat space on both sides. The metric on this setup is given by

$$\begin{aligned} ds^2_{in} &= \frac{4\pi^2}{\beta^2}\frac{dyd\bar{y}}{\sinh^2\frac{\pi}{\beta}(y+\bar{y})}, \quad ds^2_{out} = \frac{1}{\epsilon^2}dyd\bar{y}, \\ y &= \sigma + i\tau, \quad \bar{y} = \sigma - i\tau, \quad \tau \sim \tau + \beta. \end{aligned} \tag{6.42}$$

The subscript 'in' refers to the gravity zone, and 'out' refers to the flat region. The coordinates are reminiscent of those we introduced when we discussed the conformal welding problem. In the gluing process we introduced a cutoff at the circle $\sigma = -\epsilon$ and set up transparent boundary conditions on it. Some other coordinates for the inside region are:

$$\begin{aligned} &Schwarzchild\ coodrinates: \ y = \frac{\beta}{2\pi}\ln\frac{r}{\sqrt{r(r+4\pi/\beta)}} + i\tau, \quad ds^2_{in} = r(r+\frac{4\pi}{\beta})d\tau^2 + \frac{dr^2}{r(r+\frac{4\pi}{\beta})}. \\ &Poincare\ coordinates: \ x = \tanh\frac{\pi y}{\beta}, \quad ds^2_{in} = \frac{4dxd\bar{x}}{(x+\bar{x})^2}. \end{aligned} \tag{6.43}$$

In this setup one can see that finding the welding maps is trivial. To see this we simply present them and see that they work. These maps are given by

$$z = v = w = e^{2\pi y/\beta}, \quad y = \frac{\beta}{2\pi}\ln w. \tag{6.44}$$

Now the cutoff at $\sigma = -\epsilon$ is at $w = e^{-2\pi\epsilon/\beta}$ so for small ϵ we have in the inside region, $|w| < 1 - \frac{2\pi\epsilon}{\beta}$. Then the metric in two regions is expressed as:

$$ds^2_{in} = \frac{4dwd\bar{w}}{(1-|w|^2)^2}, \quad ds^2_{out} = \frac{\beta^2}{4\pi^2\epsilon^2}\frac{dwd\bar{w}}{|w|^2}. \tag{6.45}$$

Next we check that coordinate w above, produces a smooth metric for all the points on the inside and outside region. For the points with $|w| < 1 - \frac{2\pi\epsilon}{\beta}$ the ds^2_{in} in 6.45 is smooth and for the points with $|w| > 1 - \frac{2\pi\epsilon}{\beta}$, ds^2_{out} in 6.45 is smooth. So the thing that really matters is to check that these metrics are smooth and continuous at the boundary. So we write:

$$\begin{aligned} ds^2_{in}\Big|_{|w|=1-\frac{2\pi\epsilon}{\beta}} &= \frac{4dwd\bar{w}}{(1-(1+\frac{4\pi^2\epsilon^2}{\beta^2}-\frac{4\pi\epsilon}{\beta}))^2} = \frac{4dwd\bar{w}}{(\frac{4\pi\epsilon}{\beta}-\frac{4\pi^2\epsilon^2}{\beta^2})^2} \\ &= \frac{\beta^2}{4\pi^2\epsilon^2}\frac{dwd\bar{w}}{|w|^2}\Big|_{|w|=1-\frac{2\pi\epsilon}{\beta}} = ds^2_{out}\Big|_{|w|=1-\frac{2\pi\epsilon}{\beta}}. \end{aligned} \tag{6.46}$$

This checking assures us that the welding maps we chose indeed cover the entire glued geometry. This was expected because here we are considering gluing a hyperbolic disk on the inside to the punctured complex plane on the outside and the resulting geometry is expected to be the complex plane by the welding maps chosen from the usual coordinates on each of the two sides. If we consider more complicated setups, as we shall see later, the resulting welding maps won't be trivial.

One can also solve the dilaton's equation of motion using the metric in the inside region to find

$$\phi = \frac{2\pi\phi_r}{\beta}\frac{1+|w|^2}{1-|w|^2} = \frac{2\pi\phi_r}{\beta}\frac{1}{\tanh\frac{2\pi\sigma}{\beta}}, \tag{6.47}$$

where $\phi_b = \frac{\phi_r}{\epsilon}$ at the boundary. We can set $\epsilon = 0$ by rescaling the exterior coordinate by ϵ so that $ds^2_{out} = dyd\bar{y}$.

The next step is to use the QES prescription to calculate the entropy. As we said above, the geometry has two parts, a gravitational AdS_2 region inside and flat region on the outside. The same CFT lives in both regions with the transparent boundary condition in between. The interval that we consider for calculating the entanglement entropy is an interval $[-a, b],\ a, b > 0$ at a given time. In order to do the calculation, following the QES method we introduce cosmic branes at the conical singularities which are the endpoints of the intervals that we want to calculate the entanglement on. Then we calculate the area for these branes and add it to the CFT entropy. Then the S_{gen} is given by:

$$S_{gen} = Area\ terms + S_{CFT}([-a,b]) = S_0 + \phi(-a) + S_{CFT}([-a,b]). \tag{6.48}$$

To calculate the CFT entropy, we can use the results of calculations of Cardy and Calabrese in $2D$ CFTs. By looking at 6.45, one can realize that in the both inside and outside regions by a suitable change of coordinates $z = f(w, \bar{w})$ from the complex plane with coordinates $w, \bar{w}$ one arrives at 6.45. This means that we have:

$$ds^2 = \frac{\partial f}{\partial w}\frac{\partial \bar{f}}{\partial \bar{w}}dwd\bar{w} = \Omega^{-2}(w,\bar{w})dwd\bar{w} = \Omega^{-2}ds^2_{flat}, \tag{6.49}$$

where ds^2 is the metric in the original geometry given by 6.45 in the inside and outside, and ds^2_{flat} is the flat metric on the complex plane with coordinates $w, \bar{w}$. The equation above is basically the statement of the fact that every $2-$dimensional metric is conformally flat. The value of product of the derivatives of f in this equation can be easily read from 5.27, by inverting the coefficients of $dwd\bar{w}$. Now we remember 3.29 which allowed us to relate the results of entanglement entropy of a CFT in the complex plane to different conformally flat geometries.

Then we do a calculation reminiscent to 3.30 for the interval $[w_1, w_2]$:

$$\begin{aligned} tr(\rho^n_{[w_1,w_2]})_{AdS_2+Minkoswki} &= \Big(\frac{\partial f}{\partial w}|_{w=w_1}\Big)^{-h_n}\Big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=\bar{w}_1}\Big)^{-h_n}\Big(\frac{\partial f}{\partial w}|_{w=w_2}\Big)^{-h_n}\Big(\frac{\partial \bar{f}}{\partial \bar{w}}|_{\bar{w}=\bar{w}_2}\Big)^{-h_n} \\ &\quad \langle \mathcal{T}_n(w_1,\bar{w}_1)\tilde{\mathcal{T}}_n(w,\bar{w}_2)\rangle_{\mathbb{C}} \\ &= \Omega^{2h_n}(w_1,\bar{w}_1)\Omega^{2h_n}(w_2,\bar{w}_2)\langle \mathcal{T}_n(w_1,\bar{w}_1)\tilde{\mathcal{T}}_n(w_2,\bar{w}_2)\rangle_{\mathbb{C}} \\ &= \Bigg(\Omega(w_1,\bar{w}_1)\Omega(w_2,\bar{w}_2)\Bigg)^{\frac{c}{12}(n-1/n)} c_n\Bigg(\frac{|w_1-w_2|^2}{\epsilon_{1,UV}\epsilon_{2,UV}}\Bigg)^{-\frac{c}{12}(n-1/n)} \\ &= c_n\Bigg(\frac{|w_1-w_2|^2}{\epsilon_{1,UV}\epsilon_{2,UV}\Omega(w_1,\bar{w}_1)\Omega(w_2,\bar{w}_2)}\Bigg)^{-\frac{c}{12}(n-1/n)}. \end{aligned} \tag{6.50}$$

One has to note that in the above calculation we have the original complex plane with coordinates $w, \bar{w}$ and our geometry with coordinates $f(w, \bar{w})$, which means that we had to use the inverted version of the 3.29, and this is the reason for negative exponents in for the derivatives of fs in the first line. The ϵs are the usual UV cutoffs in the calculation of entropy for a CFT. Now we use 6.50 to calculate the entropy:

$$\begin{aligned} S^{(n)}_{CFT}([w_1,w_2]) &= \frac{1}{1-n}\ln\left(\Bigg(\frac{|w_1-w_2|^2}{\epsilon_{1,UV}\epsilon_{2,UV}\Omega(w_1,\bar{w}_1)\Omega(w_2,\bar{w}_2)}\Bigg)^{-\frac{c}{12}(n-1/n)}\right) + c'_n \\ S_{CFT}([w_1,w_2]) &= \lim_{n\to 1} S^{(n)}_{CFT}([w_1,w_2]) = \frac{c}{6}\ln\left(\frac{|w_1-w_2|^2}{\epsilon_{1,UV}\epsilon_{2,UV}\Omega(w_1,\bar{w}_1)\Omega(w_2,\bar{w}_2)}\right). \end{aligned} \tag{6.51}$$

Finally using the map $w = e^{2\pi y/\beta}$ and the conformal factors in 6.45 we can write:

$$S_{CFT}([-a,b]) = \frac{c}{6}\ln\left(\frac{2\beta\,\sinh^2\left(\frac{\pi}{\beta}(a+b)\right)}{\epsilon_{a,UV}\epsilon_{b,UV}\pi\,\sinh\left(\frac{2\pi a}{\beta}\right)}\right). \tag{6.52}$$

Now we plug in the dilaton 6.47 in 6.48 and write:

$$\boxed{S_{gen}([-a,b]) = S_0 + \frac{2\pi\phi_r}{\beta}\frac{1}{\tanh\frac{2\pi a}{\beta}} + \frac{c}{6}\ln\left(\frac{2\beta\,\sinh^2\left(\frac{\pi}{\beta}(a+b)\right)}{\pi\epsilon\,\sinh\left(\frac{2\pi a}{\beta}\right)}\right).} \tag{6.53}$$

The UV divergence $\epsilon_{a,UV}$ was absorbed into S_0 and the one on the outside was dropped. As a final step in the calculation we have to extrimize S_{gen} over a to find the quantum extremal surface

$$\boxed{\partial_a S_{gen} = 0 \;\;\rightarrow\;\;\; \sinh\left(\frac{2\pi a}{\beta}\right) = \frac{12\pi\phi_r}{\beta c}\frac{\sinh\left(\frac{\pi}{\beta}(a+b)\right)}{\sinh\left(\frac{\pi}{\beta}(a-b)\right)}.} \tag{6.54}$$

This is a cubic equation for $e^{2\pi a/\beta}$. For asymptotic regions of $b \geq \frac{\beta}{2\pi}$ and $\phi_r/(\beta c) \geq 1$, the solution is

$$a \approx b + \frac{\beta}{2\pi}\ln\left(\frac{24\pi\phi_r}{\beta c}\right), \quad or \quad e^{\frac{2\pi a}{\beta}} \approx \frac{\beta c}{24\pi\phi_r}e^{-\frac{2\pi b}{\beta}}. \tag{6.55}$$

The calculations in here were restricted to one side of the black hole. In one side moving in a certain direction in Schwarzschild time t is an isometry, so the configuration is invariant under the time translation and the extremal surface at $t \neq 0$ is related by a time translation to the result above so that for an interval that starts at t_b and $\sigma_b = b$ the other point is at $t_a = t_b$ and $\sigma_a = -a$, with the a that is found in 6.54.

6.5.2 Calculations using the replica wormholes

Now we try to derive the results above directly from the replica method in the geometry. This is good practice of the welding problem. Also it gives us a hint on how the entropy derivation in QES is not dependent on holography. We use Euclidean signature in which a, b are real. We also set $\beta = 2\pi$ and reintroduce it later by dimensional analysis.

We introduce an $n-$fold cover of the Euclidean black hole, branched at a and b that we introduced above. Unlike the case in the last section in this $n-$fold geometry the welding maps are not trivial, so it is convenient to introduce different coordinates on the inside and outside. We use w, with $|w| < 1$, for the inside and $v = e^y$, with $|v| > 1$ for the outside. We again have the gluing function $\theta(\tau)$ with $w = e^{i\theta}, v = e^{i\tau}$ as in 6.25. We write

$$w = A = e^{-a}, \quad v = B = e^b. \tag{6.56}$$

The inside coordinates can be uniformized into a hyperbolic disk using the uniformization theorem in the complex analysis with the map

$$\tilde{w} = \left(\frac{w - A}{1 - Aw}\right)^{\frac{1}{n}} \tag{6.57}$$

In this new coordinates the inside region has the usual metric of the hyperbolic disk

$$ds_{in}^2 = \frac{4d\tilde{w}d\bar{\tilde{w}}}{|1 - |\tilde{w}|^2|^2}, \tag{6.58}$$

on which the calculations are the same as the usual AdS_2 and the results can be taken from there. At the boundary we define $\tilde{w} = e^{i\tilde{\theta}}$. Now that we are at AdS_2 we can write the Schwarzian equation, which relates the change of energy to flux of energy to the outside [30]

$$\frac{\phi_r}{4\pi}\partial_\tau\{e^{i\tilde{\theta}}, \tau\} = \frac{d}{d\tau}E = i(T_{yy}(i\tau) - T_{\bar{y}\bar{y}}(-i\tau)). \tag{6.59}$$

Now we translate this equation into the original coordinates w, using the Schwarzian composition identity

$$\{e^{i\tilde{\theta}}, \tau\} = \{e^{i\theta}, \tau\} + \frac{1}{2}\left(1 - \frac{1}{n^2}\right)R(\theta), \quad R(\theta) = -\frac{(1 - A^2)^2(\partial_\tau\theta)^2}{|1 - Ae^{i\theta}|^4}, \tag{6.60}$$

which gives:

$$\frac{\phi_r}{4\pi}\partial_\tau\left(\{e^{i\theta}, \tau\} + \frac{1}{2}\left(1 - \frac{1}{n^2}\right)R(\theta)\right) = i(T_{yy}(i\tau) - T_{\bar{y}\bar{y}}(-i\tau)). \tag{6.61}$$

The next step is to find the stress tensor on the right-hand side of 6.61. This is done by translating the coordinates to the complex plane. As we mentioned above this is possible through

the conformal welding problem. Doing the conformal welding we will have a map $G(w)$ on the inside and a map $F(v)$ on the outside as in 6.25. In this case the image of these maps is a an $n-$sheeted structure reminiscent to the structure that we have for the original calculation of Cardy and Calabrese. We get this $n-$sheeted structure because our geometry is an $n-$folded cover of the original setup of figure 22. The maps F, G result in a coordinate z in the $n-$sheeted Riemann surface that it's coordinates transform under the group $SL(2,\mathbb{C})$. So we can use this transformations to make sure that $w = A$ is mapped to $z = 0$ and $v = B$ is mapped to $z \to \infty$.
The coordinate z cover the whole $n-$sheeted Riemann surface holomorphically. This structure can be mapped to the original complex plane using $\tilde{z} = z^{1/n}$ where $\tilde{z}$ are now coordinates on $\mathbb{C}$. Now we can write:

$$T_{zz}(z) = \left(\frac{\partial \tilde{z}}{\partial z}\right) T_{\tilde{z}\tilde{z}} - \frac{c}{24\pi}\{z^{1/n}, z\} = -\frac{c}{24\pi}\{z^{1/n}, z\} = -\frac{c}{48\pi}\left(1 - \frac{1}{n^2}\right)\frac{1}{z^2}, \tag{6.62}$$

where we used the fact that on the complex plane with coordinate $\tilde{z}$, the stress energy tensor vanishes. Next we remember that $F(v) = F(e^y) = z$, so we have:

$$\begin{aligned} &Schwarzian\ composition\ :\ \{f\ o\ g, t\} = \left(\frac{\partial g}{\partial t}\right)^2 \{f, g\} + \{g, t\} \\ &\xRightarrow{t=y, g=v=e^y, f=F(e^y)} \{F(e^y), y\} = e^{2y}\{F(v), v\} + \{e^y, y\} \\ &\qquad = e^{2y}\{F(v), v\} - \frac{1}{2}. \end{aligned} \tag{6.63}$$

Then we have using 6.63:

$$\begin{aligned} T_{yy}(y) &= \left(\frac{\partial F}{\partial (e^y)} e^y\right)^2 T_{zz} - \frac{c}{24\pi}\{F(e^y), y\} \\ &= e^{2y}\left(\left(\frac{\partial F}{\partial v}\right)^2 T_{zz} - \frac{c}{24\pi}\{F(v), v\}\right) - \frac{1}{2}. \end{aligned} \tag{6.64}$$

Putting 6.61, 6.62 and 6.64 together, and reintroducing β we have:

$$\boxed{\frac{24\pi\phi_r}{c\beta}\partial_\tau\left[\{e^{i\theta(\tau)}, \tau\} + \frac{1}{2}(1 - \frac{1}{n^2})R(\theta(\tau))\right] = ie^{2i\tau}\left[-\frac{1}{2}(1 - \frac{1}{n^2})\frac{F'(e^{i\tau})^2}{F(e^{i\tau})^2} - \{F, e^{i\tau}\}\right] + c.c.\ .} \tag{6.65}$$

This is as far as this review goes for finite n. This formula is complicated because the map F depends implicitly on the gluing function $\theta(\tau)$.

Now we try to solve the equation above for the case $n \to 1$ for any β. The aim is to rederive the QES equation without holography. First we look at the case with $n = 1$. As we have seen above in 6.44, in this case the welding problem is trivial and we can set $w = v$ everywhere. A convenient choice is

$$z = F(v) = \frac{v - A}{B - v} = G(w), \quad w = v. \tag{6.66}$$

At $n=1$ any choice of A can do because different choices are related by $SL(2,\mathbb{R})$ transformations on w plane. We choose A so that as $n\to 1$, it corresponds to the position of the conical singularity. Near $n\sim 1$ we expand

$$e^{i\theta}=e^{i\tau}+e^{i\tau}i\delta\theta(\tau), \tag{6.67}$$

where $\delta\theta$ is of order $n-1$. The aim is to solve 6.65 for $\delta\theta$. First step is to find the welding map pertubatively in $n-1$. We then have from 6.41:

$$e^{2i\tau}\{F,e^{i\tau}\}=-\delta\{e^{i\tau},\tau\}_-=-(\delta\theta'''+\delta\theta')_-, \tag{6.68}$$

where the definition of the first variation was used:

$$\delta\{e^{i\theta},\tau\}:=\{e^{i\tau+i\delta\theta},\tau\}-\{e^{i\tau},\tau\}=\delta\theta'''+\delta\theta'. \tag{6.69}$$

The negative subscript means that the functions are projected onto negative-frequency Fourier modes. This projection can be expressed using the Hilbert transform, H, which is defined by the action $H.e^{im\tau}=-sgn(m)e^{im\tau}$, by

$$e^{2i\tau}\{F,e^{i\tau}\}=-\frac{1}{2}(1+H)(\delta\theta'''+\delta\theta'). \tag{6.70}$$

It is obvious that the expression above is a projection because for any positive frequency $e^{ik\tau}, k>0$ one has:

$$\frac{1}{2}(1+H)e^{ik\tau}=\frac{1}{2}e^{ik\tau}-\frac{1}{2}e^{ik\tau}=0. \tag{6.71}$$

And for any negative frequency mode $e^{il\tau}, l<0$ one has:

$$\frac{1}{2}(1+H)e^{il\tau}=\frac{1}{2}e^{il\tau}+\frac{1}{2}e^{il\tau}=e^{il\tau}. \tag{6.72}$$

Except for $\{F,e^{i\tau}\}$, wherever else F appears in 6.65, it is multiplied by $n-1$. so if we only keep the zeroth order $F=\frac{v-A}{B-v}$ in these cases, the expressions are already of order $n-1$. Therefore one can rewrite 6.65 up to first order in $n-1$ as:

$$\partial_\tau(\delta\theta'''+\delta\theta')+\frac{ic}{12\phi_r}H.(\delta\theta'''+\delta\theta')=(n-1)\left[\frac{c}{12\phi_r}\mathcal{F}-\partial_\tau R(\tau)\right], \tag{6.73}$$

where

$$\mathcal{F}=-i\frac{e^{2i\tau}(A-B)^2}{(e^{i\tau}-A)^2(e^{i\tau}-B)^2}+c.c.\,. \tag{6.74}$$

Due to the Hilbert transform, 6.73 is nonlocal. As we mentioned before this shows that F depends non-locally on $\theta(\tau)$. Now to solve it we expand both sides in a Fourier series. If there is a Fourier series of the form $\sum_k a_k e^{ik\tau}$ for $\delta\theta$, then for each k, one has in the left hand side of 6.73:

$$lhs\ \ for\ \ each\,k:\ \ a_k(-ik^3+ik)+a_k\frac{ic}{12\phi_r}H.(-ik^3+ik). \tag{6.75}$$

This expression is automatically zero when $k=0,\pm1$. This must be satisfied for the modes on the righthand side as well. For $k=1$ to find the coefficient for this mode one integrates and set the integral equal to zero:

$$\int_0^{2\pi}d\tau e^{-i\tau}\left(\frac{c}{12\phi_r}\mathcal{F}-\partial_\tau R(\tau)\right)=0. \tag{6.76}$$

Then the integration results in the condition

$$\boxed{\frac{c}{6\phi_r}\frac{\sinh\frac{a-b}{2}}{\sinh\frac{a+b}{2}} = \frac{1}{\sinh a}.} \tag{6.77}$$

This is precisely the quantum extremal surface equation 6.54, which we found earlier. Thus one can find the QES directly from the equations of motion and the replica wormholes. Once the QES condition is imposed the rest of the Fourier modes can be found readily to confirm that $\delta\theta$ is indeed a solution.

Now we can calculate the entropy using the replica method by the formula:

$$S = -\partial_n\left(\frac{I^{tot}}{n}\right)\Big|_{n=1}. \tag{6.78}$$

We saw above that this is equivalent to finding the first order expansion of action in terms of $n-1$. It is most convenient to work in the Schwarzian theory action for gravity. This can be found the same way we found 6.61. In the uniformized hyperbolic disk coordinates we know that the gravitaional action for JT reduces to boundary term that can be written in terms of the Schwarzian [30]:

$$\begin{aligned} I_{grav} &= -S_0 - \frac{\phi_r}{2\pi}n\int_0^{2\pi} d\tau\{e^{i\tilde{\theta}(\tau)},\tau\} \\ &= -S_0 - \frac{\phi_r}{2\pi}n\int_0^{2\pi} d\tau\left(\{e^{i\theta},\tau\} + \frac{1}{2}\left(1-\frac{1}{n^2}\right)R(\theta)\right), \end{aligned} \tag{6.79}$$

where in the second line we transformed back into the original coordinates of our setup. The first term, $-S_0$ comes from the Hilbert-Einstein term in the action which in the $2D$ gravity using the Gauss-Bonnet theorem is equal to the Euler characteristic of the geometry times $-S_0$, which in our case of replica wormholes is $1\times(-S_0)$.
Now we find the first order term of taylor expansion of $-\frac{I_{grav}}{n}$ around $n=1$:

$$\begin{aligned} (n-1)\partial_n\left(\frac{-I_{grav}}{n}\right)\Big|_{n=1} &= (n-1)\left(-\frac{1}{n^2}S_0\Big|_{n=1}\right) + (n-1)\frac{\phi_r}{2\pi}\int_0^{2\pi} d\tau\partial_n\{e^{i\theta},\tau\}\big|_{n=1} \\ &+ (n-1)\frac{\phi_r}{2\pi}\int_0^{2\pi} d\tau\left(\frac{1}{n^3}R(\theta)\Big|_{n=1}\right) \\ &= (1-n)S_0 + (n-1)\frac{\phi_r}{2\pi}\int_0^{2\pi} d\tau R(\tau) \\ &+ (n-1)\frac{\phi_r}{2\pi}\int_0^{2\pi} d\tau\partial_n\{e^{i\theta},\tau\}\big|_{n=1}. \end{aligned} \tag{6.80}$$

The first term above gives the S_0 contribution to the area term. The second one is the dilaton at the branch point a:

$$\frac{\phi_r}{2\pi}\int_0^{2\pi} d\tau\, R(\tau) = -\frac{\phi_r}{tanh\, a}. \tag{6.81}$$

The third term will be canceled with the matter contribution. The leading term in the matter action is the von Neumann entropy of the CFT, as calculated by the method of Cardy and Calabrese, plus contribution from an order $(n-1)$ change in the metric

$$\ln Z_n^{mat} - n\, \ln Z_1^{mat} = -(n-1) S_{bulk\,CFT}([-a, b]) + \delta_g\, \ln Z^{mat}. \tag{6.82}$$

The matter action is evaluated on the geometry with dynamical twist point in the gravity region, so the bulk entropy includes the island I. By the equation of motion at $n = 1$, the last term above is canceled with the last term in 6.80. Then one arrives at the S_{gen} as it was found using the QES, which shows that QES is indeed valid without assumption of holography.

6.6 Information Paradox

Finally we put everything together and explain how the consideration of the islands and the replica wormholes can potentially resolve the information paradox.

6.6.1 Information Paradox in Zero Temperature

We start by considering a simple formulation of the information paradox in zero temperature. Consider the region R in the figure 23. If we consider a fixed background, the von Neumann entropy of the quantum fields on this region is infrared divergent. This is basically a Hawking-like calculation of the entropy using quantum field theory on a fixed background.

If the unitarity is valid then the state on the full cauchy slice is pure. But the AdS_2 is a quantum system with e^{S_0} states, which is the exponential of the entropy of this gravity region. This is a contradiction, because the infinite states of quantum field theory on the outside cannot be purified with the finite states in the AdS_2 region.

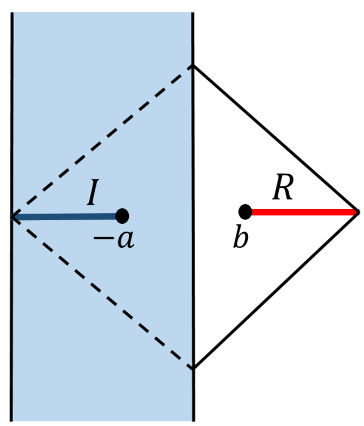


Figure 23: An information paradox at zero temperature, with AdS_2 on the left and flat space on the right. The Hawking-like calculation of matter entropy in region R is infrared divergent, but this cannot be purified by quantum gravity in AdS_2. This is resolved by including the island, I.

This contradiction can be resolved by introducing an island region as in figure 23. In the calculations above we demonstrated that the QES prescription works without the assumption of holography at least in the context of $2D$ gravity, so we can use it here. Then by taking the limit $\beta \to \infty$ of the 6.53 we have:

$$\boxed{S_{gen}(I \cup R) = S_0 + \frac{\phi_r}{a} + \frac{c}{6} \ln \frac{(a+b)^2}{a}.} \tag{6.83}$$

And setting $\partial_a S_{gen} = 0$ gives the location of the QES,

$$a = \frac{1}{2}(k + b + \sqrt{b^2 + 6bk + k^2}), \;\; k := \frac{6\phi_r}{c}. \tag{6.84}$$

Now the entropy of the radiation and island together is no longer infrared divergent and the contradiction is resolved.

6.6.2 Information Paradox in the Eternal Black Hole

As we mentioned above, for the case of AdS_2 eternal black holes, glued to the flat space from two sides, we have the information problem. The radiation entropy increases indefinitely if we one does the calculations for the Hawking saddles. However the unitary page curve in this case predicts a saturation after the page time [31]. This means that we expect a behavior like figure 24. Our aim here is to show that consideration of the islands produces this page curve

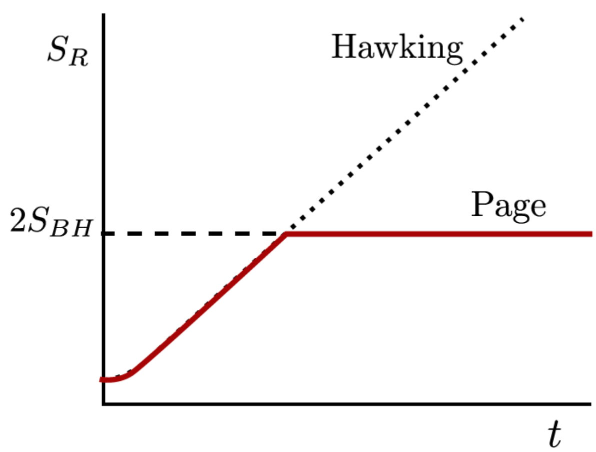


Figure 24: Page curve for the Hawking calculation for the eternal black hole and the unitary expected page curve. This figure is taken from [6].

The Lorentzian representation of the setup with the island can be seen in figure 25. Again we set $\beta = 2\pi$ and in the (σ, τ) coordinates we have

$$P_1 = (-a, t_a), \quad P_2 = (b, t_b), \quad P_3 = (-a, -t_a + i\pi), \quad P_4 = (b, -t_b + i\pi). \tag{6.85}$$

The island and radiation regions are given by

$$R = [P_4, \infty_L) \cup [P_2, \infty_R), \qquad I = [P_3, P_1]. \tag{6.86}$$

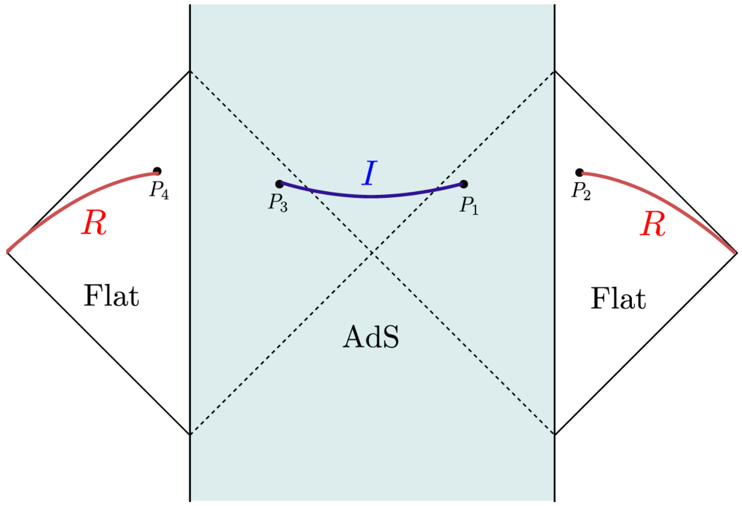


Figure 25: Eternal black hole in AdS_2, glued to Minkowski space on both sides. Hawking radiation is collected in region R, which has two disjoint components. Region I is the island.

We are considering the full system to be in pure state, so we have

$$S_{CFT}(I \cup R) = S_{CFT}([P_4, P_3] \cup [P_1, P_2]). \tag{6.87}$$

This is a two interval setup. If we consider the CFT to be a free fermion theory, then 3.36 with the consideration of the conformal factors gives the following relation for the entropy:

$$\boxed{S_{fermions}(I \cup R) = \frac{c}{3} \ln \left(\frac{2 \cosh t_a \cosh t_b |cosh(t_a - t_b) - \cosh(a+b)|}{\sinh a \cosh(\frac{a+b-t_a-t_b}{2}) \cosh(\frac{a+b+t_a+t_b}{2})} \right).} \tag{6.88}$$

Now we can calculate the generalized entropy, with the inclusion of the island:

$$\boxed{S_{gen}^{islands}(I \cup R) = 2S_0 + \frac{2\phi_r}{\tanh a} + S_{fermions}(I \cup R).} \tag{6.89}$$

Note that the area term in S_{gen} is found using the result of the single interval derived above with the consideration that we have two sides and have to multiply by 2. When there are no islands the entropy of the radiation is just given by the entropy of the CFT in the region $[P_4, P_2]$:

$$\boxed{S_{gen}^{no\ islands}(R) = S_{fermions}(R) = \frac{c}{3} \ln(2 \cosh t_b).} \tag{6.90}$$

If one looks at the extremality condition $\partial_a S_{gen}^{islands} = 0$ one can see that this equation doe not necessarily have a real solution at the early times. In particular at $t_a = t_b = 0$ one has:

$$\frac{6\phi_r}{c} \sinh(a+b) = 2 \sinh^2 a - \sinh a \cosh a \sinh(a+b). \tag{6.91}$$

The existence of real solutions for this equation depends on parameters b and ϕ_r/c. However at late times $\partial_a S_{gen}^{islands} = 0$ always has a real solution. Then the entropy using the QES prescription is given by

$$\boxed{S(R) = min\{S_{gen}^{islands}, S_{gen}^{no\ islands}\}.} \tag{6.92}$$

This gives the desired page curve of figure 24. At the early times there are no islands and 6.90 gives the desired linear increase due to combination of logarithm and exponentials. At the later times the island contribution can be approximated:

$$S_{fermions}(I \cup R) \approx 2S_{fermions}([P_1, P_2]) = \frac{c}{3} \ln \left(\frac{2|\cosh(a+b) - \cosh(t_a - t_b)|}{\sinh a} \right). \tag{6.93}$$

This agrees with the constant part of the desired page curve.

Thus we see that indeed the contribution of the replica wormholes seems to resolve the information problem and give the right page curve. The calculations done up to now in this chapter were all in 2-dimensions. But one has to realize that although the calculations were highly simplified and the figures could be drawn easier in this case, the topological argument we explained for the creation of islands due to the replica wormholes is quite general and there doesn't seem to be any reason that it does not apply to the more general cases. In fact in the next section we will try to explain how this topological argument can be used to argue the resolution of the information paradox with simple arguments.

6.6.3 A Simple but General Argument

Here we follow [8] to more conceptually explain the results of this chapter.

If we imagine that k quanta of radiation have left a black hole and assume that the total state of the black hole and the radiation is pure, with the use of the Schmidt decomposition we can write:

$$|\Psi\rangle \, \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle_B \, |i\rangle_R \, . \tag{6.94}$$

Here the Schmidt basis of the radiation is given by $\{|i\rangle_R\}_{i=1}^{k}$, and that of the interior partner in the black hole is given by $\{|\psi_i\rangle_B\}_{i=1}^{k}$.

To employ the replica trick one has to calculate the $tr(\rho_R^n)$. The density matrix of the radiation is given by

$$\rho_R = \frac{1}{k} \sum_{ij} \left({}_B\langle\psi_i|\psi_j\rangle_B \right) |j\rangle_R \, {}_R\langle i| \, . \tag{6.95}$$

and for $n = 2$

$$\rho_R^2 = \frac{1}{k^2} \sum_{ijm} \left({}_B\langle\psi_i|\psi_j\rangle_B \right) \overline{\left({}_B\langle\psi_i|\psi_m\rangle_B \right)} \, |j\rangle_R \, {}_R\langle m| \, . \tag{6.96}$$

The gravitational path integral gives orthogonal states so one expects to have

$${}_B\langle\psi_i|\psi_j\rangle_B = \delta_{ij}. \tag{6.97}$$

But when one goes on to calculate the higher powers of the radiation density matrix things become more interesting. As we mentioned before, when we replicate the geometry due to it being dynamical, other saddles may contribute to the path integral. The disconnected saddles would give simple orthogonal contributions

$$\boxed{\left({}_B \langle\psi_i|\psi_j\rangle_B\right)\overline{\left({}_B \langle\psi_i|\psi_j\rangle_B\right)} \supset disconnected = \delta_{ij},} \tag{6.98}$$

where $\supset$ implies that we are considering certain contributions to the expression. However for the connected saddles, this is not true. One can see this by an argument similar to those that we explained earlier. Due to the fact that the action is of the form $I = S_0\chi$ and $\chi \neq 0$ for the connected saddles, The path integral would give non-orthonormal contributions. For the case of $n = 2$, one has $\chi = 1$ and

$$\boxed{\left({}_B \langle\psi_i|\psi_j\rangle_B\right)\overline{\left({}_B \langle\psi_i|\psi_j\rangle_B\right)} \supset connected \approx e^{-A/4G}.} \tag{6.99}$$

So one can expect to have

$$tr(\rho_R^n) = k\frac{e^{-I_{n,disconnected}}}{k^n\mathcal{N}^n} + k^n\frac{e^{-I_{n,connected}}}{k^n\mathcal{N}^n} = k^{1-n} + e^{(1-n)A/4G}, \tag{6.100}$$

where $\mathcal{N}$ is a normalization constant and $(1-n)$ comes from the Euler characteristic χ of the nth connected geometry. At $n \to 1$ the exponential with the smaller base will dominate in the above expression. This means that one can write:

$$\boxed{S(R) = \lim_{n\to 1}\frac{1}{1-n}\ln tr(\rho_R^n) \approx min\{\ln k, A/4G\}} \tag{6.101}$$

This is exactly the QES prescribed entropy which is consistent with the page curve and the results in the last section. It also gives a conceptual explanation of the general setup and how everything fits together. At the early times k is small and the term $\ln k$ contributes giving the first rise in the page curve. At late times k is large and the area term of the black hole contributes to give the second half of the page curve.

6.7 Ensemble Interpretation

Putting together the contributions of 6.98 and 6.99 one has that

$$|{}_B \langle\psi_i|\psi_j\rangle_B|^2 = \delta_{ij} + e^{-A/4G}. \tag{6.102}$$

This is a rather odd result and seems to contradict 6.97. How does one explain this inconsistency? [8] suggests that one can view the gravitational path integral as computing averages in a statistical ensemble of microscopic theories. This theories can be parameterized by a random variable R_{ij}. For example if one assumes $\langle R_{ij}\rangle = 0$ and $\langle R_{ij}^2\rangle = 1$ and

$$\boxed{{}_B \langle\psi_i|\psi_j\rangle_B = \delta_{ij} + R_{ij}e^{-A/8G}.} \tag{6.103}$$

Then one has:

$$\begin{aligned} \langle {}_B\langle\psi_i|\psi_j\rangle_B \rangle &= \delta_{ij}, \\ \langle |{}_B\langle\psi_i|\psi_j\rangle_B|^2 \rangle &= \delta_{ij} + e^{-A/4G}. \end{aligned} \tag{6.104}$$

Thus the result would be consistent with both 6.97 and 6.102. This idea is indeed a bold one however it is still too soon to conclude that gravity is an ensemble of theories. An array of investigations and discussions are in motion to argue for and against this view but for now we conclude our discussion here.

7 Acknowledgements

We thank Behrad Taghavi, Zahra Kabiri, Amirhossein Tajdini and Amin Faraji Astaneh for discussions and explanations for the concepts that was mentioned in this thesis.

References

[1] S.W. Hawking, Particle creation by black holes, Comm. Math. Phys. 43 (1975) 199-220.

[2] Don N. Page. Information in Black Hole Radiation, hep-th/9306083, Phys. Rev. Lett. 71 (1993) 3743–3746.

[3] Calabrese, Pasquale, and Cardy, John L., “Entanglement entropy and quantum field theory”, J. Stat. Mech., 0406, P06002, (2004). [DOI], [arXiv:hep-th/0405152].

[4] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. 96, 181602, 2006, [arXiv:hep-th/0603001 [hep-th]].

[5] A. Lewkowycz, J. Maldacena, ”Generalized gravitational entropy”, arXiv:1304.4926 [hep-th]

[6] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, “Replica wormholes and the entropy of Hawking radiation,” JHEP 05 (2020), 013 [arXiv:1911.12333 [hep-th]].

[7] N. Engelhardt and A. C. Wall, “Quantum extremal surfaces: Holographic entanglement entropy beyond the classical regime,” JHEP 01 (2015), 073 [arXiv:1408.3203 [hep-th]].

[8] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, “Replica wormholes and the black hole interior,” [arXiv:1911.11977 [hep-th]].

[9] P. Townsend, black holes

[10] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press (1973).

[11] J.M. Bardeen, B. Carter, S.W. Hawking, The four laws of black hole mechanics, Comm. Math. Phys. 31 (1973). 161

[12] J. Bekenstein, Black holes and entropy, Phys. Rev. D7 (1973). 2333; Generalized second law of thermodynamics in black hole physics, Phys. Rev. D9 (1974). 3293

[13] D. Harlow, Jerusalem Lectures on Black Holes and Quantum Information. arXiv:1409.1231 [hep-th]

[14] Martin B. Plenio, quant-ph/0504163

[15] Carlson, R., Sur une classe de séries de Taylor, (Dissertation, Uppsala, Sweden, 1914).

[16] Sergey N. Solodukhin, Entanglement entropy of black holes, arXiv:1104.3712 [hep-th]

[17] H. Casini, C. D. Fosco and M. Huerta, Entanglement and alpha entropies for a massive Dirac eld in two dimensions, J. Stat. Mech. 0507, P07007, 2005, [arXiv:cond-mat/0505563 [cond-mat]].

[18] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, "Geometry of the (2+1) black hole," Phys. Rev. D 48 (1993) 1506 Erratum: [Phys. Rev. D 88 (2013) 069902] doi:10.1103/PhysRevD.48.1506, 10.1103/PhysRevD.88.069902 [gr-qc/9302012].

[19] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity", Commun.Math.Phys. 104 (1986) 207–226.

[20] T. Faulkner, "The entanglement Renyi entropies of disjoint intervals in AdS/CFT", arXiv:1303.7221 [hep-th]

[21] T. Hartman, "Entanglement Entropy at Large Central Charge", arXiv:1303.6955 [hep-th]

[22] K. Krasnov, "Holography and Riemann surfaces," Adv. Theor. Math. Phys. 4, 929 (2000) [hep-th/0005106].

[23] G. W. Gibbons, S. W. Hawking and , Phys. Rev. D 15, 2752 (1977).

[24] T. Hartman, "Lectures on Quantum Gravity and Black Holes".

[25] D. V. Fursaev, S. N. Solodukhin, "On the Description of the Riemannian Geometry in the Presence of Conical Defects", arXiv:hep-th/9501127 [hep-th].

[26] D. V. Fursaev, A. Patrushev, S. N. Solodukhin, "Distributional Geometry of Squashed Cones", arXiv:1306.4000 [hep-th]

[27] J. Camps, W. R. Kelly, "Generalized gravitational entropy without replica symmetry," arXiv:1412.4093 [hep-th].

[28] X. Dong, "The gravity dual of Rényi entropy," Nature Commun. 7, 12472 (2016) [arXiv:1601.06788 [hep-th]].

[29] M. Blau, Lecture Notes on General Relativity.

[30] arXiv:1606.03438 [hep-th]

[31] A. Almheiri, R. Mahajan and J. Maldacena, Islands outside the horizon, 2019, [arXiv:1910.11077 [hep-th]].