بنام خدا

پاسخ نامه تمرین سری دوم :

سوال اول

الف)

$$|n+\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i\varphi} \end{pmatrix} \qquad |n-\rangle = \begin{pmatrix} -\sin\theta/2 \, e^{-i\varphi} \\ \cos\theta/2 \end{pmatrix}$$

ب)

$$R_n(\theta) = e^{-i\theta S_n/\hbar} = \cos\theta/2 \, I - i\sin\theta/2 \, \vec{\sigma}\cdot\hat{n}$$

$$\hat{n} = (\cos\varphi, \sin\varphi, 0)$$

$$\vec{\sigma}\cdot\hat{n} = \cos\varphi \, \sigma_x + \sin\varphi \, \sigma_y = \frac{(\sigma_+ + \sigma_-)}{2}\cos\varphi + \left(\frac{\sigma_+ - \sigma_-}{2i}\right)\sin\varphi$$

$$= \frac{1}{2}\left[(\cos\varphi - i\sin\varphi)\sigma_+ + (\cos\varphi + i\sin\varphi)\sigma_-\right]$$

$$= \frac{1}{2}\left[e^{-i\varphi}\sigma_+ + e^{i\varphi}\sigma_-\right]$$

$$\longrightarrow R_n(\theta) = \cos\theta/2 \, I - i\sin\theta/2 \, \frac{1}{2}\left(e^{-i\varphi}\sigma_+ + e^{i\varphi}\sigma_-\right)$$

$$R_n(\theta)|z+\rangle = \cos\theta/2 \, |z+\rangle \underbrace{-i}_{e^{-i\pi/2}}\sin\theta/2 \, e^{i\varphi} \underbrace{\frac{\sigma_-}{2}|z+\rangle}_{|z-\rangle} = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \, e^{i(\varphi-\pi/2)} \end{pmatrix}$$

$$R_n(\theta)|z-\rangle = \cos\theta/2 \, |z-\rangle \underbrace{-i}_{e^{i\pi/2}}\sin\theta/2 \, e^{-i\varphi} \underbrace{\frac{\sigma_+}{2}|z-\rangle}_{|z+\rangle} = \begin{pmatrix} -\sin\theta/2 \, e^{-i(\varphi-\pi/2)} \\ \cos\theta/2 \end{pmatrix}$$

$$R_{n(\varphi)}(\theta)|z+\rangle = |n+(\theta, \varphi-\pi/2)\rangle \qquad R_{n(\varphi)}(\theta)|z-\rangle = |n-(\theta, \varphi-\pi/2)\rangle$$

ج) $R_{n(\varphi)}(\theta)$: عملگر دوران در فضای اسپینورها

$$R_n(\theta = 2\pi) = -I$$

$\longrightarrow$ حالت ذرات اسپین ۱/۲ پس ۲π دوران به منفی خود می‌رود.

سوال دوم

a) $\{A\cdot\sigma , B\cdot\sigma\} = 2\, A\cdot B$

$$\{A\cdot\sigma , B\cdot\sigma\} = \{A_i\sigma_i , B_j\sigma_j\} = A_iB_j \underbrace{\{\sigma_i , \sigma_j\}}_{2\delta_{ij}}$$

$$= 2\, A_iB_i = 2\, A\cdot B$$

b) $(\sigma\cdot A)(\sigma\cdot B) = A\cdot B + i\, \sigma\cdot(A\times B)$

$$(\sigma\cdot A)(\sigma\cdot B) = (\sigma_i A_i)(\sigma_j B_j) = A_iB_j\ \sigma_i\sigma_j$$

$$\sigma_i\sigma_j = \begin{cases} I & i=j \\ i\epsilon_{ijk}\sigma_k & i\neq j \end{cases}$$

$$[\sigma_i , \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

$$\{\sigma_i , \sigma_j\} = \delta_{ij} \longrightarrow i\neq j : \quad \sigma_i\sigma_j = \sigma_j\sigma_i \longrightarrow \begin{array}{l} 2\sigma_i\sigma_j = 2i\epsilon_{ijk}\sigma_k \\ \sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k \end{array}$$

$$\longrightarrow \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

$$\longrightarrow (\sigma\cdot A)(\sigma\cdot B) = A_iB_j\,(\delta_{ij} + i\epsilon_{ijk}\sigma_k)$$

$$= A_iB_j\,\delta_{ij} + i\epsilon_{ijk}\,A_iB_j\,\sigma_k$$

$$= A\cdot B + i\,(A\times B)_k\,\sigma_k = A\cdot B + i\,(A\times B)\cdot\sigma$$

$Q_{(2)}$:

c) $e^{i\alpha\sigma_x}\sigma_z e^{-i\alpha\sigma_x} = (\cos\alpha + i\sigma_x\sin\alpha)\sigma_z(\cos\alpha - i\sigma_x\sin\alpha)$

$= \sigma_z\cos^2\alpha + \sigma_x\sigma_z\sigma_x\sin^2\alpha + i[\sigma_x, \sigma_z]\sin\alpha$

از طرفی : $\sigma_x\sigma_z = -\sigma_z\sigma_x$, $\sigma_x^2 = I$. $[\sigma_x, \sigma_z] = -2i\sigma_y$

$\Rightarrow e^{i\alpha\sigma_x}\sigma_z e^{-i\alpha\sigma_x} = \sigma_z\cos^2\alpha - \sigma_z\sigma_x^2\sin^2\alpha + 2\sigma_y\sin\alpha\cos\alpha$

$= \sigma_z(\cos^2\alpha - \sin^2\alpha) + \sigma_y(2\sin\alpha\cos\alpha)$

$\Rightarrow e^{i\alpha\sigma_x}\sigma_z e^{-i\alpha\sigma_x} = \sigma_z\cos(2\alpha) + \sigma_y\sin(2\alpha)$

سوال سوم

$$H=\frac{\alpha}{\hbar^2}(S_x^2+S_y^2)+\frac{\alpha}{\hbar}S_z=\frac{\alpha}{\hbar^2}(S^2-S_z^2)+\frac{\alpha}{\hbar}S_z$$

$$=\frac{\alpha}{\hbar^2}S^2-\frac{\alpha}{\hbar^2}(S_z^2-\hbar S_z)\ ,\quad H|\psi\rangle=E|\psi\rangle$$

$$|\psi\rangle=\sum_{S,m}A_{Sm}|S,m\rangle\ \rightarrow\ \sum_{Sm}A_{Sm}\left(\frac{\alpha}{\hbar^2}S(S+1)\hbar^2-\frac{\alpha}{\hbar^2}(m^2\hbar^2-m\hbar^2)\right)|S,m\rangle$$

$$=\sum_{S,m}A_{Sm}\left(S(S+1)-m(m-1)\right)\alpha|S,m\rangle=\sum_{S,m}EA_{Sm}|S,m\rangle$$

$$\rightarrow\ A_{Sm}\left(E-\alpha(S(S+1)-m(m-1))\right)=0$$

$$\rightarrow\ E_{Sm}=\alpha\left[S(S+1)-m(m-1)\right]\ ,\quad |E_{Sm}\rangle=|S,m\rangle$$

$$S=5/2\ \rightarrow\ E_m=\alpha\left(\frac{35}{4}-m(m-1)\right)$$

$$m=5/2\ \rightarrow\ E_{5/2}=\alpha\left(\frac{35}{4}-\frac{15}{4}\right)=5\alpha$$

$$m=3/2\ \rightarrow\ E_{3/2}=\alpha\left(\frac{35}{4}-\frac{3}{4}\right)=8\alpha$$

$$m=1/2\ \rightarrow\ E_{1/2}=\alpha\left(\frac{35}{4}+\frac{1}{4}\right)=9\alpha$$

$$m=-1/2\ \rightarrow\ E_{-1/2}=\alpha\left(\frac{35}{4}-3/4\right)=8\alpha$$

$$m=-3/2\ \rightarrow\ E_{-3/2}=\alpha\left(\frac{35}{4}-\frac{15}{4}\right)=5\alpha$$

$$m=-5/2\ \rightarrow\ E_{-5/2}=\alpha\left(\frac{35}{4}-\frac{35}{4}\right)=0$$

$\rightarrow$ $E=5\alpha$ تبهگن : $|5/2,5/2\rangle$, $|5/2,-3/2\rangle$

$E=8\alpha$ تبهگن : $|5/2,3/2\rangle$, $|5/2,-1/2\rangle$

بقیه حالات انرژی تبهگنی ندارند.

$Q_{(4)}$:

$$\Psi = kr(\cos\phi\sin\theta + \sin\phi\sin\theta + 2\cos\theta)e^{-\alpha r}$$

Angular Part:

$$\Psi(\theta,\phi) = k'(\cos\phi\sin\theta + \sin\phi\sin\theta + 2\cos\theta)$$

$$\left.\begin{aligned} &\Rightarrow k'^2\int_0^{\pi} d\theta \int_0^{2\pi} (\cos\phi\sin\theta + \sin\phi\sin\theta + 2\cos\theta)^2 \sin\theta\, d\phi = 1 \\ &\text{[illegible]}: \cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}) \\ &\qquad \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}) \end{aligned}\right\} \Longrightarrow$$

$$\Psi(\theta,\phi) = k'\left[\frac{1}{2}(e^{i\phi} + e^{-i\phi})\sin\theta + \frac{1}{2i}(e^{i\phi} - e^{-i\phi})\sin\theta + \cos 2\theta\right]$$

$$= k'\left[-\frac{1}{2}(1-i)\sqrt{\frac{8\pi}{3}}\,Y_1^1 + \frac{1}{2}(1+i)\sqrt{\frac{8\pi}{3}}\,Y_1^{-1} + 2\sqrt{\frac{4\pi}{3}}\,Y_1^0\right]$$

$$\Rightarrow k'^2\left[\frac{1}{2}\,\frac{8\pi}{3} + \frac{1}{2}\,\frac{8\pi}{3} + 4\cdot\frac{4\pi}{3}\right] = 1 \quad \Rightarrow \quad k' = \sqrt{\frac{1}{8\pi}}$$

$$\Rightarrow \boxed{\Psi(\theta,\phi) = \sqrt{\frac{1}{8\pi}}\left[-\frac{1}{2}(1-i)\sqrt{\frac{8\pi}{3}}\,Y_1^1 + \frac{1}{2}(1+i)\sqrt{\frac{8\pi}{3}}\,Y_1^{-1} + 2\sqrt{\frac{4\pi}{3}}\,Y_1^0\right]}$$

[illegible])

$$\sqrt{\langle L^2\rangle} = \sqrt{l(l+1)}\,\hbar = \sqrt{2}\,\hbar$$

ب) $\langle \psi | L_z | \psi \rangle = k'^2 \Big[\frac{1}{2}\frac{8\pi}{3}\hbar (Y_1^1)^2 + \frac{1}{2}\frac{8\pi}{3}(-\hbar)(Y_1^{-1})^2 + 4 \cdot \frac{4\pi}{3}(0)(Y_1^0$

$$= \frac{1}{8\pi}\left[\frac{1}{2}\frac{8\pi}{3}\hbar - \frac{1}{2}\frac{8\pi}{3}\hbar\right] = 0$$

ب) $P = |\langle L_z = +\hbar \,|\, \psi(\theta,\phi)\rangle|^2$

$$= \frac{1}{8\pi} \times \frac{1}{2} \times \frac{8\pi}{3} = \frac{1}{6}$$

سؤال پنجم

$$\vec{\mu} = \mu_B \vec{\sigma}$$

$$\hat{H} = \begin{cases} -\vec{\mu}\cdot\vec{B} & \vec{B} = B\hat{z} \quad 0<t<T \\ -\vec{\mu}\cdot\vec{B} & \vec{B} = B\hat{y} \quad T<t<2T \end{cases}$$

$$\hat{H} = -\mu_B B \begin{cases} \sigma_z & 0<t<T \\ \sigma_y & T<t<2T \end{cases}$$

حالت اولیه سیستم : $|\psi(0)\rangle = |x+\rangle = \dfrac{|z+\rangle + |z-\rangle}{\sqrt{2}}$

$0<t<T \qquad \hat{H} = -\mu_B B \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$: [illegible]

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{+i\mu_B B t/\hbar\, \sigma_z}|\psi(0)\rangle$$

$$= \left(\cos\left(\frac{\mu_B B t}{\hbar}\right) I + i\sin\left(\frac{\mu_B B t}{\hbar}\right)\sigma_z \right) \underbrace{|\psi(0)\rangle}_{|x+\rangle}$$

$$= \cos\left(\frac{\mu_B B t}{\hbar}\right)|x+\rangle + i\sin\left(\frac{\mu_B B t}{\hbar}\right)|x-\rangle$$

$$t = T \longrightarrow |\psi(T)\rangle = \cos\left(\frac{\mu B t}{\hbar}\right)|x+\rangle + i\sin\left(\frac{\mu B t}{\hbar}\right)|x-\rangle$$

$$T<t<2T \qquad |\psi(t)\rangle = e^{-i\hat{H}(t-T)/\hbar}|\psi(T)\rangle = e^{+i\mu B\frac{(t-T)}{\hbar}\sigma_y}|\psi(T)\rangle$$

$$= \left(\cos\left(\frac{\mu B(t-T)}{\hbar}\right) I + i\sin\left(\frac{\mu B(t-T)}{\hbar}\right)\sigma_y \right)|\psi(T)\rangle$$

$$= \cos\left(\frac{\mu B(t-T)}{\hbar}\right)|\psi(T)\rangle + i\sin\left(\frac{\mu B(t-T)}{\hbar}\right)\sigma_y|\psi(T)\rangle$$

$$\sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$\sigma_y|x+\rangle = \begin{pmatrix} & -i \\ i & \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ i \end{pmatrix} = -i|x-\rangle$$

$$\sigma_y|x-\rangle = \begin{pmatrix} & -i \\ i & \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ i \end{pmatrix} = i|x+\rangle$$

$$|\psi(۲T)\rangle = \cos\left(\frac{\mu B T}{\hbar}\right)|\psi(T)\rangle + i\sin\left(\frac{\mu B T}{\hbar}\right)\left\{\cos\left(\frac{\mu B T}{\hbar}\right)\underbrace{\sigma_y|x+\rangle}_{-i|x-\rangle} + i\sin\left(\frac{\mu B T}{\hbar}\right)\underbrace{\sigma_y|x-\rangle}_{i|x+\rangle}\right\}$$

$$= \left\{\cos^۲\left(\frac{\mu B T}{\hbar}\right) - i\sin^۲\left(\frac{\mu B T}{\hbar}\right)\right\}|x+\rangle + \left\{i\sin\left(\frac{\mu B T}{\hbar}\right)\cos\left(\frac{\mu B T}{\hbar}\right) + \sin\left(\frac{\mu B T}{\hbar}\right)\cos\left(\frac{\mu B T}{\hbar}\right)\right\}|x-\rangle$$

$$P(\text{[illegible]}\ |x+\rangle\ \text{[illegible]}) = |\langle x+|\psi(۲T)\rangle|^۲$$

$$= \left|\underbrace{\cos^۲\left(\frac{\mu B T}{\hbar}\right)}_{\frac{1+\cos\left(\frac{۲\mu B T}{\hbar}\right)}{۲}} - i\underbrace{\sin^۲\left(\frac{\mu B T}{\hbar}\right)}_{\frac{1-\cos\left(\frac{۲\mu B T}{\hbar}\right)}{۲}}\right|^۲$$

$$= \frac{1}{۴}\left\{\left(1+\cos\left(\frac{۲\mu B T}{\hbar}\right)\right)^۲ + \left(1-\cos\left(\frac{۲\mu B T}{\hbar}\right)\right)^۲\right\}$$

$$= \frac{1}{۴}\left\{۲ + ۲\cos^۲\left(۲\frac{\mu B T}{\hbar}\right)\right\} = \frac{1}{۲}\left\{1+\cos^۲\left(\frac{۲\mu B T}{\hbar}\right)\right\}$$

سوال ششم :

$$[J_x J_y, J_z] + [J_x, J_y J_z] = J_x \overbrace{[J_y, J_z]}^{i\hbar J_x} + \overbrace{[J_x, J_z]}^{-i\hbar J_y} J_y + \overbrace{[J_x, J_y]}^{i\hbar J_z} J_z + J_y \overbrace{[J_x, J_z]}^{-i\hbar J_y}$$

$$= i\hbar J_x^2 - i\hbar J_y^2 + i\hbar J_z^2 - i\hbar J_y^2 = \boxed{i\hbar\left(J_x^2 - 2J_y^2 + J_z^2\right)}$$

سؤال هفتم :

الف) $|0,0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$

$$\begin{cases} |1,1\rangle = |+,+\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle + |-,+\rangle\right) \\ |1,-1\rangle = |-,-\rangle \end{cases}$$

ب)

$$|\psi\rangle = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\left(|1,0\rangle + |00\rangle\right) + \frac{1}{\sqrt{2}}|1,-1\rangle$$

$$= \frac{1}{2}|1,0\rangle + \frac{1}{\sqrt{2}}|1,-1\rangle + \frac{1}{\sqrt{2}}|00\rangle$$

از اندازه گیری تکانه ی کل S^2 مقدار صفر بدست آید. مقدار S_z کل هم حتما صفر است.

$$P(\ell=0) = |\langle 0,0|\psi\rangle|^2 = \frac{1}{2}$$

ممکن است از اندازه گیری تکانه ی کل (S^2) مقدار ۱ بدست آید، که در این صورت مقدار S_z می تواند $+\hbar$، ۰ و $-\hbar$ باشد.

$$P(\ell=1) = P(\ell=1, m=1) + P(\ell=1, m=0) + P(\ell=1, m=-1)$$

$$P(\ell=1, m=0) = |\langle 1,0|\psi\rangle|^2 = 1/4$$

$$P(\ell=1, m=1) = |\langle 1,1|\psi\rangle|^2 = 0$$

$$P(\ell=1, -1) = |\langle 1,-1|\psi\rangle|^2 = 1/4$$

$$P(\ell=1) = 0 + 1/4 + 1/4 = 1/2$$

سوال هشتم :

$$P = \begin{pmatrix} \langle +,+|P|+,+\rangle & \langle +,+|P|+,-\rangle & \langle +,+|P|-,+\rangle & \langle +,+|P|-,-\rangle \\ \langle +,-|P|+,+\rangle & \langle +,-|P|+,-\rangle & \langle +,-|P|-,+\rangle & \langle +,-|P|-,-\rangle \\ \langle -,+|P|+,+\rangle & \langle -,+|P|+,-\rangle & \langle -,+|P|-,+\rangle & \langle -,+|P|-,-\rangle \\ \langle -,-|P|+,+\rangle & \langle -,-|P|+,-\rangle & \langle -,-|P|-,+\rangle & \langle -,-|P|-,-\rangle \end{pmatrix}$$

در واقع رابطه‌ی بالا با فرض زیر به دست می‌آید.

$$|+,+\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |+,-\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |-,+\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-,-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|\alpha\rangle = \alpha_1|+\rangle + \alpha_2|-\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$|\beta\rangle = \beta_1|+\rangle + \beta_2|-\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\ \alpha_2 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \end{pmatrix}$$

$$|\beta\rangle \otimes |\alpha\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \otimes \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_2\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_2 \end{pmatrix}$$

$$P|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_2\beta_1 \\ \alpha_1\beta_2 \\ \alpha_2\beta_2 \end{pmatrix} = |\beta\rangle \otimes |\alpha\rangle$$

ادامه ی سوال هشتم:

ج)

$$\sigma_0 \otimes \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_3 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3 = 2\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 2P$$