(1

الف)

$$|j_1-j_2| \leq j \leq |j_1+j_2| \to 3/2 \leq j \leq 5/2$$

$$\to j \in \{3/2, 5/2\}$$

ب)

$$H = a + b/2\left((L+S)^2 - L^2 - S^2\right) + cL^2 = a + b/2\, J^2 + (c-b/2)L^2 - b/2\, S^2$$

$$\to H|J,m;L,s\rangle = \left(a + j(j+1)\hbar^2 b/2 + (c-b/2)\,l(l+1)\hbar^2 - b/2\, s(s+1)\hbar^2\right)|J,m;L,s\rangle$$

$$\to E_j = \left(a + \frac{b\hbar^2}{2} j(j+1) + 6\hbar^2(c-b/2) - \frac{3b}{8}\hbar^2\right)$$

(4

$$H = \frac{\lambda}{2}\left(2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}\right) + \frac{eB}{mc}\left(S_{1z} - S_{2z}\right)$$

ماتریس هامیلتونی در پایه $|m_1, m_2\rangle$

$$H = \begin{bmatrix} \langle ++|H|++\rangle & \langle ++|H|+-\rangle & \langle ++|H|-+\rangle & \langle ++|H|--\rangle \\ \langle +-|H|++\rangle & \langle +-|H|+-\rangle & \langle +-|H|-+\rangle & \langle +-|H|--\rangle \\ \langle -+|H|++\rangle & \langle -+|H|+-\rangle & \langle -+|H|-+\rangle & \langle -+|H|--\rangle \\ \langle --|H|++\rangle & \langle --|H|+-\rangle & \langle --|H|-+\rangle & \langle --|H|--\rangle \end{bmatrix}$$

$$H = \begin{bmatrix} \lambda/2\,\hbar^2 & 0 & 0 & 0 \\ 0 & -\lambda/4\,\hbar^2 + \frac{eB\hbar}{mc} & \frac{\lambda}{2}\hbar^2 & 0 \\ 0 & \lambda/2\,\hbar^2 & -\lambda/4\,\hbar^2 - \frac{eB\hbar}{mc} & 0 \\ 0 & 0 & 0 & \lambda/4\,\hbar^2 \end{bmatrix}$$

با قطری کردن این ماتریس ویژه مقادیر و ویژه بردارهای آن را می یابیم. $\det(H-IE)=0$

$E_1=\lambda/_4\hbar^2 \quad , |\psi_1\rangle=|++\rangle$

$E_2=-\frac{\lambda}{4}\hbar^2+\sqrt{(\frac{eB\hbar}{mc})^2+\frac{\lambda^2}{4}\hbar^4} \quad , |\psi_2\rangle=\dfrac{\frac{\lambda\hbar^2}{2}|+-\rangle+\left(\frac{-eB\hbar}{mc}+\sqrt{\frac{(eB\hbar)^2}{m^2c^2}+\frac{\lambda^2}{4}\hbar^4}\right)|-+\rangle}{\sqrt{((\frac{\lambda}{2}\hbar^2)^2+\left(-\frac{eB\hbar}{mc}+\sqrt{(\frac{eB\hbar}{mc})^2+\frac{\lambda^2}{4}\hbar^4}\right)^2}}$

$E_3=\frac{-\lambda}{4}\hbar^2-\sqrt{(\frac{eB\hbar}{mc})^2+\frac{\lambda^2\hbar^4}{4}} \quad ; |\psi_3\rangle=\dfrac{\frac{-eB\hbar}{mc}|-+\rangle+\left(\frac{\lambda}{2}\hbar^2+\sqrt{(\frac{eB\hbar}{mc})^2+\frac{\lambda^2}{4}\hbar^4}\right)|+-\rangle}{\sqrt{\frac{e^2B^2\hbar^2}{m^2c^2}+\left(\frac{\lambda}{2}\hbar^2+\sqrt{(\frac{eB\hbar}{mc})^2+\frac{\lambda^2}{4}\hbar^4}\right)^2}}$

$E_4=\lambda/_4\hbar^2 \quad ; |\psi_4\rangle=|--\rangle$

ب)

$H=\frac{eB}{mc}(S_{1z}-S_{2z}) \rightarrow H|m_1,m_2\rangle=\frac{eB}{mc}(m_1-m_2)|m_1,m_2\rangle$

ج)

$H=\lambda S_1\cdot S_2=\frac{\lambda}{2}((S_1+S_2)^2-S_1^2-S_2^2) \rightarrow H|S_T,m_T;s_1,s_2\rangle=\frac{\lambda}{2}(S_T(S_T+1)\hbar^2-\frac{3}{2}\hbar^2)|S_Tm_T;s_1,s_2\rangle$

Q2):

$$\left.\begin{array}{l}\hat{\vec{S}}_{12} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 \\ \hat{\vec{S}}_{34} = \hat{\vec{S}}_3 + \hat{\vec{S}}_4\end{array}\right\} \Rightarrow \hat{\vec{S}} = \hat{\vec{S}}_{12} + \hat{\vec{S}}_{34}$$

برای $S_{12}=0,1$ ، $S_{34}=0,1$ در کل 16 حالت داریم که:

1) $S_{12}=0$, $S_{34}=0$ $\Rightarrow$ حالت ممکن $S=0$ $\Rightarrow |s,m\rangle = |0,0\rangle$

2) $S_{12}=1$, $S_{34}=0$ $\Rightarrow$ حالت ممکن $S=1$ $\Rightarrow$ $|s,m\rangle = |1,\pm1\rangle$
$|s,m\rangle = |1,0\rangle$

3) $S_{12}=0$, $S_{34}=1$ $\Rightarrow$ حالت ممکن $S=1$ $\Rightarrow$ $|s,m\rangle = |1,\pm1\rangle$
$|s,m\rangle = |1,0\rangle$

4) $S_{12}=1$, $S_{34}=1$ $\Rightarrow$ حالت ممکن $S=0,1,2$ $\Rightarrow$ $|0,0\rangle$
$|1,\pm1\rangle$
$|1,0\rangle$
$|2,\pm2\rangle$
$|2,\pm1\rangle$
$|2,0\rangle$

در کل وقتی چهار ذره به یکدیگر کوپل می‌شوند، اسپین کل می‌تواند مقادیر 0، 1، 2 را بگیرد که برای مقدار 0 دوبار، و برای مقدار 1 سه بار و برای مقدار 2 یک بار اتفاق می‌افتد.

Q(3)

الف) S_{12} : $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ $\quad S_1 = 1/2 \,,\; S_2 = 1/2$

$$S_{12} \otimes S_3 \longrightarrow \begin{cases} S_{12} = 0 & 0 \otimes \frac{1}{2} = \frac{1}{2} \quad \left(S = \frac{1}{2}, S_{12} = 0\right) \\ S_{12} = 1 & 1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \quad \left(S = \frac{1}{2}, S_{12} = 1\right) \left(S = \frac{3}{2}, S_{12} = 1\right) \end{cases}$$

$S = 3/2 \qquad m = -3/2, -1/2, 1/2, 3/2$

$|3/2, 3/2\rangle = |+++\rangle \qquad |3/2, -3/2\rangle = |---\rangle$

$J_- \downarrow \qquad J_- = J_{1-} + J_{2-} + J_{3-}$

$$|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|-++\rangle + |+-+\rangle + |++-\rangle\right)$$

J_+

$$|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|+--\rangle + |-+-\rangle + |--+\rangle\right)$$

$$|S_{12} = 0, S = \tfrac{1}{2}, \tfrac{1}{2}\rangle = \alpha\,|+-+\rangle + \beta\,|-++\rangle$$

↲ ۲ ذره در حالت $+\frac{1}{2}$

و یک ذره در حالت $-\frac{1}{2}$

ذره ی اول و دوم باید خلاف هم باشند.

این حالت باید بر $|3/2, 1/2\rangle$ عمود باشد

و $\alpha^2 + \beta^2 = 1$

$$\alpha = -\beta = \frac{1}{\sqrt{2}}$$

$$\longrightarrow |S_{12} = 0, S = \tfrac{1}{2}, \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|+-+\rangle - |-++\rangle\right)$$

$J_- \downarrow$

$$|S_{12} = 0, S = \tfrac{1}{2}, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(-|+--\rangle - |-+-\rangle\right)$$

$$|S_{12}=1, S=\tfrac{1}{2}, \tfrac{1}{2}\rangle = \alpha|++-\rangle + \beta|+-+\rangle + \gamma|-++\rangle$$

$$\alpha^2+\beta^2+\gamma^2=1$$

$$\langle S_{12}=0, S=\tfrac{1}{2}, \tfrac{1}{2}| S_{12}=1, \tfrac{1}{2}, \tfrac{1}{2}\rangle = 0 \longrightarrow \beta=\gamma$$

$$\langle S_{12}=0, S=\tfrac{1}{2}, \tfrac{1}{2}| S=\tfrac{3}{2}, \tfrac{1}{2}\rangle = 0 = \frac{1}{\sqrt{3}}(\alpha+\beta+\gamma)$$

$$\longrightarrow \begin{cases} \beta=\gamma \\ \alpha+\beta+\gamma=0 \longrightarrow \alpha+2\beta=0 \longrightarrow \alpha=-2\beta \end{cases}$$

$$\alpha^2+\beta^2+\gamma^2 = 4\beta^2+\beta^2+\beta^2 = 6\beta^2=1 \longrightarrow \beta=\frac{1}{\sqrt{6}}=\gamma$$

$$\alpha=-\frac{2}{\sqrt{6}}$$

$$|S_{12}=1, S=\tfrac{1}{2}, \tfrac{1}{2}\rangle = -\frac{2}{\sqrt{6}}|++-\rangle + \frac{1}{\sqrt{6}}|+-+\rangle + \frac{1}{\sqrt{6}}|-++\rangle$$

$$J_- \Big\downarrow$$

$$|S_{12}=1, S=\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\frac{1}{\sqrt{6}}|-+-\rangle + \frac{2}{\sqrt{6}}|--+\rangle - \frac{1}{\sqrt{6}}|+--\rangle$$

Q(3):

چون سه ذره با یکدیگر متفاوت اند می توان هامیلتونی سیستم را به شکل زیر نوشت.

$$\hat{H} = -\left(\frac{\varepsilon_0}{\hbar^2}\right)\left(\hat{\vec{S}}_1+\hat{\vec{S}}_2\right)\cdot\hat{\vec{S}}_3 = -\frac{\varepsilon_0}{\hbar^2}\,\hat{\vec{S}}_{12}\cdot\hat{\vec{S}}_3 = -\frac{\varepsilon_0}{2\hbar^2}\left(\hat{S}^2-\hat{S}_{12}^2-\hat{S}_3^2\right)$$

از طرفی: $\hat{\vec{S}}_{12}\cdot\hat{\vec{S}}_3 = \frac{1}{2}\left[\left(\hat{\vec{S}}_{12}+\hat{\vec{S}}_3\right)^2-\hat{\vec{S}}_{12}^2-\hat{\vec{S}}_3^2\right)$ $\Rightarrow$

$$H\,|S_{12}, S, m\rangle = -\frac{\varepsilon_0}{2\hbar^2}\left(\hat{S}^2-\hat{S}_{12}^2-\hat{S}_3^2\right)|S_{12}, S, m\rangle$$

$$\overset{(S_3=\frac{1}{2})}{=} -\frac{\varepsilon_0}{2}\left[S(S+1)-S_{12}(S_{12}+1)-\frac{3}{4}\right]|S_{12}, S, m\rangle$$

$$\Rightarrow \boxed{E_{S_{12}, S} = -\frac{\varepsilon_0}{2}\left[S(S+1)-S_{12}(S_{12}+1)-\frac{3}{4}\right]}$$

و چون به m وابستگی نیست داریم

$$E_{1,\frac{3}{2}} = -\frac{\varepsilon_0}{2} \xrightarrow{\text{برای چهار حالت}} |1,\tfrac{3}{2},\pm\tfrac{3}{2}\rangle,\ |1,\tfrac{3}{2},\pm\tfrac{1}{2}\rangle$$

$$E_{0,\frac{1}{2}} = 0 \xrightarrow{\text{برای دو حالت}} |0,\tfrac{1}{2},\pm\tfrac{1}{2}\rangle$$

$$E_{1,\frac{1}{2}} = \varepsilon_0 \xrightarrow{\text{برای دو حالت}} |1,\tfrac{1}{2},\pm\tfrac{1}{2}\rangle$$

سوال پنجم:

$j_1=2 \quad j_2=1$

$2\otimes 1 = 3 \oplus 2 \oplus 1$

$|J,m;j_1,j_2\rangle := |J,m\rangle \qquad |j_1,m_1;j_2,m_2\rangle := |m_1,m_2\rangle$

$|3,3\rangle = |m_1=2, m_2=1\rangle \xrightarrow{J_-} |3,2\rangle = \frac{\sqrt{2}}{\sqrt{3}}|1,1\rangle + \frac{1}{\sqrt{3}}|2,0\rangle$

$\xrightarrow{J_-} |3,1\rangle = \frac{\sqrt{6}}{\sqrt{15}}|0,1\rangle + \frac{\sqrt{8}}{\sqrt{15}}|1,0\rangle + \frac{1}{\sqrt{15}}|2,-1\rangle$

$\xrightarrow{J_-} |3,0\rangle = \frac{1}{\sqrt{5}}|-1,1\rangle + \frac{\sqrt{3}}{\sqrt{5}}|0,0\rangle + \frac{1}{\sqrt{5}}|1,-1\rangle$

$|3,-1\rangle = \frac{1}{\sqrt{15}}|-2,1\rangle + \frac{\sqrt{8}}{\sqrt{15}}|-1,0\rangle + \frac{\sqrt{6}}{\sqrt{15}}|0,-1\rangle$

$|3,-2\rangle = \frac{1}{\sqrt{3}}|-2,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|-1,-1\rangle$

$|3,-3\rangle = |-2,-1\rangle$

برای $J=2$: $|j=2,m=2\rangle$ تعریف می‌کنیم که بر $|j=3,m=2\rangle$ عمود باشد:

$|2,2\rangle = -\frac{1}{\sqrt{3}}|1,1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2,0\rangle$

$\xrightarrow{J_-} |2,1\rangle = \frac{1}{\sqrt{3}}|2,-1\rangle + \frac{1}{\sqrt{6}}|1,0\rangle - \frac{1}{\sqrt{2}}|0,1\rangle$

$\xrightarrow{J_-} |2,0\rangle = \frac{1}{\sqrt{2}}|1,-1\rangle - \frac{1}{\sqrt{2}}|-1,0\rangle$

$|2,-1\rangle = \frac{1}{\sqrt{3}}|-2,1\rangle + \frac{1}{\sqrt{6}}|-1,0\rangle - \frac{1}{\sqrt{2}}|0,-1\rangle$

$|2,-2\rangle = \frac{\sqrt{2}}{\sqrt{3}}|-2,0\rangle - \frac{1}{\sqrt{3}}|-1\ -1\rangle$

برای $J=1$. تعریف می‌کنیم $|j=1,m=1\rangle$ را که بر $|j=2,m=1\rangle$ و $|j=3,m=1\rangle$ عمود باشد.

$|j=1,m=1\rangle = -\frac{1}{\sqrt{10}}|0,1\rangle + \sqrt{\frac{3}{10}}|1,0\rangle - \sqrt{\frac{3}{5}}|2,-1\rangle$

$|1,0\rangle = -\sqrt{\frac{3}{10}}|1,-1\rangle + \sqrt{\frac{2}{5}}|0,0\rangle - \sqrt{\frac{3}{10}}|-1,0\rangle \qquad |1,-1\rangle = \frac{-1}{\sqrt{10}}|0,-1\rangle + \sqrt{\frac{3}{10}}|-1,0\rangle - \sqrt{3/5}\,|-2,1\rangle$

سؤال ششم

j_1, j_2

$$|j,m;j_1,j_2\rangle = \sum_{m_1',m_2'} C(j,m,m_1',m_2')\ |j_1,m_1;j_2,m_2\rangle$$

$$\xrightarrow{J_-} \hbar\sqrt{j(j+1)-m(m-1)}\ |j,m-1\rangle = \sum_{m_1',m_2'} C(j,m,m_1',m_2')\Big\{\hbar\sqrt{j_1(j_1+1)-m_1'(m_1'-1)}$$

$$|j_1,m_1'-1,j_2,m_2'\rangle + \hbar\sqrt{j_2(j_2+1)-m_2'(m_2'-1)}\ |j_1,m_1',j_2,m_2'-1\rangle\Big\}$$

$$\xrightarrow{\langle j_1,m_1,j_2,m_2|} \sqrt{j(j+1)-m(m-1)}\ C(j,m-1,m_1,m_2) =$$

$$\sqrt{j_1(j_1+1)-m_1(m_1+1)}\ C(j,m,m_1+1,m_2) + \sqrt{j_2(j_2+1)-m_2(m_2+1)}\ C(j,m,m_1,m_2+1)$$

$J_1 = l$

$J_2 = 1/2$

$J = l+1/2$

$$\langle m-1/2, 1/2\,|\,l+1/2, m\rangle = ?$$

$$C(\underset{j}{l+1/2}, \underset{m}{m}; \underset{m_1}{m-1/2}, \underset{m_2}{1/2}) = ?$$

از رابطه ی بالا استفاده می کنیم :

$$\sqrt{(l+1/2)(l+1/2+1)-(m+1)(m)}\ C(j=l+1/2, m, m-1/2, 1/2)$$

$$= \sqrt{(l+1/2)(l+1/2+1)-(m-1/2)(m-1/2+1)}\ C(j=l+1/2, m+1, m+1/2, 1/2)$$

$$+\ 0$$

$$\longrightarrow \sqrt{\underbrace{(l+1/2+m+1)}_{l+m+3/2}(l+1/2-m)}\ C(j=l+1/2, m, m-1/2, 1/2)$$

$$= \sqrt{\underbrace{(l+m-1/2+1)}_{l+m+1/2}(l-m+1/2)}\ C(j=l+1/2, m+1, m+1/2, 1/2)$$

$$C(j=l+1/2, m; m-1/2, 1/2) = \sqrt{\frac{l+m+1/2}{l+m+3/2}}\; C(j=l+1/2, m; m+1/2, 1/2)$$

$$= \sqrt{\frac{l+m+1/2}{l+\cancel{m+3/2}}}\sqrt{\frac{l+\cancel{m+3/2}}{l+m+5/2}}\; C(j=l+1/2, m; m+3/2, 1/2)$$

$$= \dots = \sqrt{\frac{l+m+1/2}{l+\cancel{m+3/2}}}\sqrt{\frac{l+\cancel{m+3/2}}{l+\cancel{m+5/2}}}\cdots\sqrt{\frac{l+l}{l+l+1}}\; \cancelto{1}{C(j=l+1/2, m; l, 1/2)}$$

$$= \sqrt{\frac{l+m+1/2}{2l+1}}$$

احتمال اینکه اسپین الکترون در حالت $m_s=1/2$ باشد وقتی $j=l+1/2$, m

$$= |C(j=l+1/2, m; m-1/2, 1/2)|^2 = \frac{l+m+1/2}{2l+1}$$

سوال هفتم:

تابع موج دو ذره ی اسپین ۱/۲ در حالت $S=0$:
$$|\psi\rangle = \frac{|z+, z-\rangle - |z-, z+\rangle}{\sqrt{2}}$$

$$|n+\rangle = \cos\theta/2\,|z+\rangle + e^{i\phi}\sin\theta/2\,|z-\rangle$$
$$|n-\rangle = \sin\theta/2\,|z+\rangle - e^{i\phi}\cos\theta/2\,|z-\rangle$$

$$\cos\theta/2\,|n+\rangle + \sin\theta/2\,|n-\rangle = |z+\rangle$$
$$\sin\theta/2\,|n+\rangle - \cos\theta/2\,|n-\rangle = e^{i\phi}|z-\rangle := |z-\rangle$$ فاز کلی در کوانتوم دیده نمی شود.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big\{\left(\cos\tfrac{\theta}{2}\,|n+\rangle + \sin\tfrac{\theta}{2}\,|n-\rangle\right)\left(\sin\tfrac{\theta}{2}\,|n+\rangle - \cos\tfrac{\theta}{2}\,|n-\rangle\right)$$

$$-\left(\sin\tfrac{\theta}{2}\,|n+\rangle - \cos\tfrac{\theta}{2}\,|n-\rangle\right)\left(\cos\tfrac{\theta}{2}\,|n+\rangle + \sin\tfrac{\theta}{2}\,|n-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big\{\Big(\cancel{\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\,|n+,n+\rangle} + \sin^2\tfrac{\theta}{2}\,|n-,n+\rangle - \cos^2\tfrac{\theta}{2}\,|n+,n-\rangle$$

$$- \cancel{\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\,|n-,n-\rangle}\Big) - \Big(\cancel{\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}}\,|n+,n+\rangle + \sin^2\tfrac{\theta}{2}\,|n+,n-\rangle$$

$$- \cos^2\tfrac{\theta}{2}\,|n-,n+\rangle - \cancel{\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}}\,|n-,n-\rangle\Big)\Big\}$$

$$= \frac{1}{\sqrt{2}}\Big\{-|n+,n-\rangle + |n-,n+\rangle\Big\} = -\left(\frac{|n+,n-\rangle - |n-,n+\rangle}{\sqrt{2}}\right)$$

فاز کلی

ب) $\langle\psi|\,\sigma_n \otimes \sigma_{n'}\,|\psi\rangle = ?$

$$= \frac{1}{2}\Big(\langle z+,z-| - \langle z-,z+|\Big)\left(\sigma_n\otimes\sigma_{n'}\right)\Big(|z+,z-\rangle - |z-,z+\rangle\Big)$$

$$= \frac{1}{2}\Big\{\langle z+|\sigma_n|z+\rangle\langle z-|\sigma_{n'}|z-\rangle - \langle z+|\sigma_n|z-\rangle\langle z-|\sigma_{n'}|z+\rangle$$

$$- \langle z-|\sigma_n|z+\rangle\langle z+|\sigma_{n'}|z-\rangle + \langle z-|\sigma_n|z-\rangle\langle z+|\sigma_{n'}|z+\rangle\Big\}$$

$$|z+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |z-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad \sigma_n = \begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta\\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}$$

$\longrightarrow$ $\langle z+|\sigma_n|z+\rangle = \cos\theta \qquad \langle z+|\sigma_n|z-\rangle = e^{-i\phi}\sin\theta$

$\langle z-|\sigma_n|z+\rangle = e^{i\phi}\sin\theta \qquad \langle z-|\sigma_n|z-\rangle = -\cos\theta$

$\longrightarrow$ $\langle\psi|\,\sigma_n\otimes\sigma_{n'}\,|\psi\rangle =$

$$\frac{1}{2}\left(-\cos\theta\cos\theta' - e^{i(\phi-\phi')}\sin\theta\,\sin\theta'\right) - \left(-e^{-i(\phi-\phi')}\sin\theta\,\sin\theta'\right.$$

$$\left.-\cos\theta\cos\theta'\right) = -\left(\cos\theta\cos\theta' + \underbrace{\cos(\phi-\phi')}_{\cos\phi\cos\phi' + \sin\phi\sin\phi'}\sin\theta\sin\theta'\right)$$

$$= -(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\cdot(\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta') = -\hat{n}\cdot\hat{n}'$$