به نام خدا

پاسخنامه تمرین سری پنجم :

سوال اول

$$H_0 = \frac{p^2}{2m} + V(x) \qquad V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{otherwise} \end{cases}$$

طیف H_0

$$\varphi_n^{(0)} = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad E_n^{(0)} = \frac{\pi^2\hbar^2 n^2}{2mL^2}$$

$$H' = V_0 \frac{x}{L}$$

تغییر انرژی تا مرتبه اول برای اختلال خواهیم داشت $= \langle \varphi_n^{(0)} | H' | \varphi_n^{(0)} \rangle$

$$\langle \varphi_n^{(0)} | H' | \varphi_n^{(0)} \rangle = \frac{2}{L}\int_0^L \sin^2\frac{n\pi x}{L}\left(V_0\frac{x}{L}\right)dx = \frac{2V_0}{L^2}\int_0^L dx\, x \sin^2\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2V_0}{L^2}\int_0^L dx\, x\left(\frac{1-\cos\frac{2n\pi x}{L}}{2}\right) = \frac{V_0}{L^2}\int_0^L dx\, x = \frac{V_0}{L^2}\frac{L^2}{2} = \frac{V_0}{2}$$

سه تراز اول انرژی : $n=1,\ n=2,\ n=3$

$$E_1 = E_1^{(0)} + \frac{V_0}{2} = \frac{\pi^2\hbar^2}{2mL^2} + \frac{V_0}{2}$$

$$E_2 = E_2^{(0)} + \frac{V_0}{2} = \frac{4\pi^2\hbar^2}{2mL^2} + \frac{V_0}{2}$$

$$E_3 = E_3^{(0)} + \frac{V_0}{2} = \frac{9\pi^2\hbar^2}{2mL^2} + \frac{V_0}{2}$$

سوال دوم :

$$H = \begin{pmatrix} 1 & \epsilon & 0 \\ \epsilon & 2 & 0 \\ 0 & 0 & 3-\epsilon \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}}_{H_0} + \underbrace{\begin{pmatrix} 0 & \epsilon & \\ \epsilon & 0 & \\ & & -\epsilon \end{pmatrix}}_{H_1}$$

ویژه مقادیر و ویژه بردارها

$$E_1^{(0)} = 1 \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_2^{(0)} = 2 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$E_3^{(0)} = 3 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

تصحیح مرتبه اول اختلال برای انرژی : $\lambda\epsilon_1 = \langle\varphi_i|H_1|\varphi_i\rangle = -\epsilon\,\delta_{i3}$

تصحیح مرتبه دوم اختلال : $$\lambda^2\epsilon_2^{(n)} = \sum_{m\neq n} \frac{|\langle\varphi_m|H'|\varphi_n\rangle|^2}{E_n^{(0)}-E_m^{(0)}}$$

$$\lambda^2\epsilon_2^{(1)} = \frac{|\langle\varphi_2|\overset{\epsilon}{\cancel{H'}}|\varphi_1\rangle|^2}{1-2} + \frac{|\langle\varphi_3|\overset{0}{\cancel{H'}}|\varphi_1\rangle|^2}{1-3} = -\epsilon^2$$

$$\lambda^2\epsilon_2^{(2)} = \frac{|\langle\varphi_1|\overset{\epsilon}{\cancel{H'}}|\varphi_2\rangle|^2}{2-1} + \frac{|\langle\varphi_3|\overset{0}{\cancel{H'}}|\varphi_2\rangle|^2}{2-3} = \epsilon^2$$

$$\lambda^2\epsilon_2^{(3)} = \frac{|\langle\varphi_1|\overset{0}{\cancel{H'}}|\varphi_3\rangle|^2}{3-1} + \frac{|\langle\varphi_2|\overset{0}{\cancel{H'}}|\varphi_3\rangle|^2}{3-2} = 0$$

$$\begin{cases} E_1 = 1-\epsilon^2 \\ E_2 = 2+\epsilon^2 \\ E_3 = 3-\epsilon \end{cases}$$

طیف H تا مرتبه دوم اختلال.

$$H = \begin{pmatrix} 1 & \epsilon & \\ \epsilon & 2 & \\ & & 3-\epsilon \end{pmatrix} \xrightarrow{\text{طیف}} \quad E = 3-\epsilon = E_3 \ \checkmark$$

$$\det\left[\begin{pmatrix} 1 & \epsilon \\ \epsilon & 2 \end{pmatrix} - \lambda I\right] = 0 \longrightarrow (1-\lambda)(2-\lambda) - \epsilon^2 = 0$$

$$\longrightarrow \lambda^2 - 3\lambda + 2 - \epsilon^2 = 0 \qquad \lambda = \frac{3 \pm \sqrt{9-8+4\epsilon^2}}{2}$$

$$= \frac{3}{2} \pm \frac{1}{2}\sqrt{1+4\epsilon^2}$$

$$= \frac{3}{2} \pm \frac{1}{2}\left(1+\frac{4\epsilon^2}{2}\right)$$

$$\longrightarrow \lambda_+ = E_+ = \frac{3}{2} + \frac{1}{2} + \frac{2\epsilon^2}{2} = 2+\epsilon^2 \ \checkmark = E_2$$

$$\lambda_- = E_- = \frac{3}{2} - \frac{1}{2} - \epsilon^2 = 1-\epsilon^2 \ \checkmark = E_1$$

سؤال سوم

عملگرهای خلق و فنا به صورت زیر است.

$$a=\sqrt{\frac{m\omega}{2\hbar}}\,x+i\,\frac{P_x}{\sqrt{2m\omega\hbar}} \qquad b=\sqrt{\frac{m\omega}{2\hbar}}\,y+i\,\frac{P_y}{\sqrt{2m\omega\hbar}}$$

$$a^{+}=\sqrt{\frac{m\omega}{2\hbar}}\,x-i\,\frac{P_x}{\sqrt{2m\omega\hbar}} \qquad b^{+}=\sqrt{\frac{m\omega}{2\hbar}}\,y-i\,\frac{P_y}{\sqrt{2m\omega\hbar}}$$

$$H=\hbar\omega\left(a^{+}a+b^{+}b+\tfrac{1}{2}+\tfrac{1}{2}\right)=\hbar\omega\left(a^{+}a+b^{+}b+1\right)$$

$$x=\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^{+}) \quad , \quad y=\sqrt{\frac{\hbar}{2m\omega}}\,(b+b^{+})$$

$$E_{n,m}=\hbar\omega\,(n+m+1) \quad , \quad |\psi_{n,m}\rangle=|n,m\rangle$$

الف)

$$E^{(1)}_{0,0}=\langle 0,0|V|0,0\rangle=2\lambda\,\langle 0|x|0\rangle\,\langle 0|y|0\rangle\times\frac{\hbar}{2m\omega}$$

$$=\frac{\lambda\hbar}{m\omega}\langle 0|a+a^{+}|0\rangle\,\langle 0|b+b^{+}|0\rangle=0$$

در این حالت واگنی دوتایی داریم.

$$E^{(0)}_{0,1}=E^{(0)}_{1,0}=2\hbar\omega$$

$$V_1^{(1)}=\begin{pmatrix}\langle 0,1|V|0,1\rangle & \langle 0,1|V|1,0\rangle\\ \langle 1,0|V|0,1\rangle & \langle 1,0|V|1,0\rangle\end{pmatrix}$$

$$\langle 0,1|V|0,1\rangle=\langle 1,0|V|1,0\rangle=0$$

$$\langle 0,1|V|1,0\rangle=\frac{\lambda\hbar}{m\omega}\langle 0|x|1\rangle\,\langle 1|y|0\rangle=\frac{\lambda\hbar}{m\omega}\langle 0|a+a^{+}|1\rangle\,\langle 1|b+b^{+}|0\rangle$$

$$=\frac{\lambda\hbar}{m\omega}\langle 0|0\rangle\,\langle 1|1\rangle=\frac{\lambda\hbar}{m\omega}$$

$$\langle 1,0|V|0,1\rangle=\frac{\lambda\hbar}{m\omega}$$

$$V_1^{(1)}=\begin{pmatrix}0 & \frac{\lambda\hbar}{m\omega}\\ \frac{\lambda\hbar}{m\omega} & 0\end{pmatrix}\longrightarrow \det\begin{pmatrix}-E_1^{(1)} & \frac{\lambda\hbar}{m\omega}\\ \frac{\lambda\hbar}{m\omega} & -E_1^{(1)}\end{pmatrix}=0$$

$$\boxed{E_1^{(1)}=\pm\frac{\lambda\hbar}{m\omega}}$$

$$E_n^{(2)} = \sum_{k\neq 0} \frac{|\langle \psi_k | V | \psi_n \rangle|^2}{E_n^{\circ} - E_k^{\circ}}$$

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$$E_{0,0}^{(2)} = \sum_{\substack{k\neq 0 \\ k'\neq 0}} \frac{|\langle k,k' | V | 0,0 \rangle|^2}{E_{0,0}^{(0)} - E_{k,k'}^{(0)}}$$

$$\langle k,k' | xy | 0,0 \rangle = \langle k | x | 0 \rangle \langle k' | y | 0 \rangle = \frac{\hbar}{2m\omega} \langle k | a + a^{\dagger} | 0 \rangle \langle k' | b + b^{\dagger} | 0 \rangle$$

$$= \frac{\hbar}{2m\omega} \delta_{k,1} \delta_{k',1}$$

$$E_{0,0}^{(2)} = (2\lambda)^2 \left(\frac{\hbar}{2m\omega}\right)^2 \sum_{k,k'} \frac{\delta_{k,1} \delta_{k',1}}{\hbar\omega - \hbar\omega(k+k'+1)} = \frac{\lambda^2\hbar^2}{m^2\omega^2} \cdot \left(\frac{1}{-2\hbar\omega}\right) = -\frac{\lambda^2 \hbar}{m^2\omega^3}$$

ج) تغییر متغیر:

$$x = \frac{X+Y}{\sqrt{2}} \quad , \quad y = \frac{X-Y}{\sqrt{2}}$$

$$X = \frac{x+y}{\sqrt{2}} \quad , \quad Y = \frac{x-y}{\sqrt{2}}$$

مشابهاً P_X ، P_Y را هم بسازیم.

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x} \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x} \frac{\partial}{\partial Y} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y} \right)$$

$$\frac{\partial^2}{\partial x^2} = \frac{1}{2} \left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y} \right) \left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y} \right) = \frac{1}{2} \frac{\partial^2}{\partial X^2} + \frac{1}{2} \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial X \partial Y}$$

$$\frac{\partial}{\partial y} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y} \right) \rightarrow \frac{\partial^2}{\partial y^2} = \frac{1}{2} \frac{\partial^2}{\partial X^2} + \frac{1}{2} \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial X \partial Y}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \quad , \quad x^2 + y^2 = X^2 + Y^2 \quad , \quad 2xy = X^2 - Y^2$$

با جایگذاری این مقادیر داریم:

$$H = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} m\omega^2 (x^2 + y^2) + 2\lambda xy$$

$$= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) + \frac{1}{2} m\omega^2 (X^2 + Y^2) + \lambda (X^2 - Y^2)$$

$$= \left[\frac{1}{2m} P_X^2 + \frac{1}{2} m \left(\omega^2 + \frac{2\lambda}{m}\right) X^2 \right] + \left[\frac{1}{2m} P_Y^2 + \frac{1}{2} m \left(\omega^2 - \frac{2\lambda}{m}\right) Y^2 \right]$$

$$H = \frac{P_x^2}{2m} + \frac{1}{2} m\omega_1^2 x^2 + \frac{P_y^2}{2m} + \frac{1}{2} m\omega_2^2 y^2$$

$$\omega_1 = \sqrt{\omega^2 + \frac{2\lambda}{m}} \qquad \omega_2 = \sqrt{\omega^2 - \frac{2\lambda}{m}}$$

$$\longrightarrow E_{n,m} = \hbar\omega_1\left(n+\frac{1}{2}\right) + \hbar\omega_2\left(m+\frac{1}{2}\right)$$

برای حالت پایه $n=0$, m . کافی است جواب را تا مرتبه‌ی دوم نسبت λ بنویسیم. $\lambda \ll 1$

$$E_{0,0} = \frac{\hbar}{2}(\omega_1+\omega_2) = \frac{\hbar\omega}{2}\left(\sqrt{1+\frac{2\lambda}{m\omega^2}} + \sqrt{1-\frac{2\lambda}{m\omega^2}}\right)$$

$$= \frac{\hbar\omega}{2}\left[\left(1+\frac{1}{2}\left(\frac{2\lambda}{m\omega^2}\right) - \frac{1}{8}\left(\frac{2\lambda}{m\omega^2}\right)^2\right) + \left(1-\frac{1}{2}\left(\frac{2\lambda}{m\omega^2}\right) - \frac{1}{8}\left(\frac{2\lambda}{m\omega^2}\right)^2\right) + O(\lambda^3)\right]$$

$$= \frac{\hbar\omega}{2}\left(2 - \frac{\lambda^2}{m^2\omega^4}\right) + O(\lambda^3) = \hbar\omega - \frac{\lambda^2\hbar}{m^2\omega^3} + O(\lambda^3)$$

جواب مرتبه‌ی اول صفر است و مرتبه‌ی دوم $\left(-\frac{\lambda^2\hbar}{m^2\omega^3}\right)$ چون جواب قسمت ج است.

سؤال چهارم :

$$H_0 = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) \qquad H_0|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle$$

$$V = \lambda \sin kX \qquad \lambda \ll 1$$

اختلال غیرتبهگن

$$\left.\begin{array}{l} E_n^{(0)} = \hbar\omega(n+1/2) \\ |\psi_n\rangle = |n\rangle \end{array}\right. \xrightarrow{\text{حالت پایه}} \left\{\begin{array}{l} E_0^{(0)} = \frac{\hbar\omega}{2} \\ |0\rangle \end{array}\right.$$

پس در اعمال V :

$$E_0 = E_0^{(0)} + \lambda\, \epsilon_1 \qquad \epsilon_1 = \langle 0|\sin kX|0\rangle$$

$$|0'\rangle = |0\rangle + \lambda \sum_{n=1}^{\infty} \frac{\langle n|\sin kX|0\rangle}{\underbrace{E_0^{(0)} - E_n^{(0)}}_{-\hbar\omega n}} |n\rangle = |0\rangle - \frac{\lambda}{\hbar\omega} \sum_{n=1}^{\infty} \frac{\langle n|\sin kX|0\rangle}{n} |n\rangle$$

$$\sin kX = \mathrm{Im}\left(e^{ikX}\right) \longrightarrow \langle n|\sin kX|0\rangle = \mathrm{Im}\left(\langle n|e^{ikX}|0\rangle\right)$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) \qquad = \mathrm{Im}\left(\langle n|e^{ik\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger+a)}|0\rangle\right)$$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2} \longrightarrow e^{\alpha(a^\dagger+a)} = e^{\alpha a^\dagger} e^{\alpha a} e^{-\alpha^2 \overbrace{[a^\dagger, a]}^{-1}/2}$$

$$\longrightarrow \langle n|\sin kX|0\rangle = \mathrm{Im}\left(e^{\alpha^2/2} \langle n|e^{\alpha a^\dagger} e^{\alpha a}|0\rangle\right) \qquad \alpha = ik\sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle n|e^{\alpha a^\dagger} e^{\alpha a}|0\rangle = \langle n|e^{\alpha a^\dagger} \sum_{i=0}^{\infty} \frac{(\alpha a)^i}{i!}|0\rangle = \langle n|e^{\alpha a^\dagger}|0\rangle$$

$$= \langle n|\sum_{i=0}^{\infty} \frac{(\alpha a^\dagger)^i}{i!}|0\rangle = \langle n|\sum_{i=0}^{\infty} \frac{\alpha^i}{i!}\sqrt{i!}\,|i\rangle$$

$$= \sum_{i=0}^{\infty} \frac{\alpha^i}{\sqrt{i!}} \underbrace{\langle n|i\rangle}_{\delta_{ni}} = \frac{\alpha^n}{\sqrt{n!}}$$

$$\langle n|\sin kX|0\rangle = \mathrm{Im}\left(e^{-\frac{\hbar k^2}{4m\omega}} \frac{\left(ik\sqrt{\frac{\hbar}{2m\omega}}\right)^n}{\sqrt{n!}}\right) = \begin{cases} 0 & n = \text{زوج} \\ e^{-\frac{\hbar k^2}{4m\omega}} \dfrac{\left(k\sqrt{\frac{\hbar}{2m\omega}}\right)^n}{\sqrt{n!}} (-1)^{\frac{n-1}{2}} & n = \text{فرد} \end{cases}$$

$$E_1 = \langle 0 | \sin kX | 0 \rangle = 0 \longrightarrow E_0 = \frac{\hbar\omega}{2}$$

$$|0'\rangle = |0\rangle - \frac{\lambda}{\hbar\omega} \sum_{n=1,3,5,\dots} \frac{e^{-\frac{\hbar k^2}{2m\omega}} \left(k\sqrt{\frac{\hbar}{2m\omega}}\right)^n}{\sqrt{n!}\; n} (-1)^{\frac{n-1}{2}} |n\rangle$$

$n = 2l+1$

$l = 0, 1, \dots$

$$|0'\rangle = |0\rangle - \frac{\lambda}{\hbar\omega} \sum_{l=0}^{\infty} \frac{e^{-\frac{\hbar k^2}{2m\omega}} \left(k\sqrt{\frac{\hbar}{2m\omega}}\right)^{2l+1}}{\sqrt{(2l+1)!}\; (2l+1)} (-1)^l |2l+1\rangle$$